

Impacts of Network Topology on the Performance of a Distributed Algorithm Solving Linear Equations

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Abstract—Recently a distributed algorithm has been proposed for multi-agent networks to solve a system of linear algebraic equations, by assuming each agent only knows part of the system and is able to communicate with nearest neighbors to update their local solutions. This paper investigates how the network topology impacts exponential convergence of the proposed algorithm. It is found that networks with higher mean degree, smaller diameter, and homogeneous degree distribution tend to achieve faster convergence. Both analytical and numerical results are provided.

I. INTRODUCTION

A major goal in studying networked systems is to understand the impact of network topology within the context of the application of interest, from epidemic spreading [2] to synchronization [3], controllability [4]–[6], observability [7], flocking [8] and consensus [9], [10].

Recently, Mou *et al.* proposed a network-based distributed algorithm to solve for x in the linear equation $\mathbf{A}x = b$ [11], [12]. In this algorithm it is assumed that each agent is located in a communication network and has partial knowledge of \mathbf{A} and b . Under mild conditions on the connectivity of the underlying network, all the agents' states (or local solutions) converge to the exact solution $x = \mathbf{A}^{-1}b$.

The proposed algorithm in [12] is distributed, applicable for all linear equations as long as they have solutions, works for time-varying networks, converges exponentially fast, operates asynchronously, and does not involve any small step-size. Note that many other algorithms formulate the problem of solving $\mathbf{A}x = b$ as a distributed optimization problem or a least square problem. But those algorithms are less efficient than the algorithm proposed in [12] in many aspects. See Ref. [13] for detailed discussions. The aim of this paper is to further characterize the relation between its exponential convergence and the network topology. The main contribution of this work is an analytical bound that

Extended arXiv version with proof is available [1].

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connects the convergence rate of the algorithm to the network topology and the linear equation. Both theoretical and numerical results show that networks with higher mean degree, smaller diameter, and homogeneous degree distributions tend to speed up this distributed algorithm.

The following notation is used throughout the paper. The ℓ^2 -norm is denoted as $\|\cdot\|$. Matrices are denoted by upper case letters in bold such as \mathbf{A} and \mathbf{P} . A partition of a matrix is denoted by an upper case letter with a subscript, i.e. A_i is a partition of matrix \mathbf{A} , which can also be a row vector. Vectors are denoted by lower case italic letters, such as x, y, z . A network or graph is denoted as $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the node (or vertex) set and \mathcal{E} is the link (or edge) set. The network topology is represented by the adjacency matrix $\mathcal{A} = \{\alpha_{ij}\}$ of the network. This paper is organized as follows. The network-based distributed algorithm is briefly presented in Section II. The theory of how the network topology impacts the algorithm performance is presented in Section III. The main proof is presented in Section IV. Finally, the conclusion is presented in Section V.

II. A DISTRIBUTED ALGORITHM FOR SOLVING LINEAR EQUATIONS

Consider a system of linear algebraic equations

$$\mathbf{A}x = b, \quad (1)$$

which has a unique solution x^* . Here $\mathbf{A} \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The partition of the matrix \mathbf{A} is defined as $\mathbf{A} = \text{col}\{A_1, A_2, \dots, A_m\}$, where $\text{col}\{\cdot\}$ is an operator that stacks elements into a column, $A_i \in \mathbb{R}^{n_i \times n}$, and the partition of the vector b is defined as $b = [b_1, b_2, \dots, b_m]^T$, $b_i \in \mathbb{R}^{n_i}$, where $\sum_{i=1}^m n_i = n$. Assume that the entire system (\mathbf{A}, b) is unavailable to a single agent; instead different partitions of the system $(A_i^{n_i \times n}, b_i^{n_i})$ are available to different agents. In this paper we consider the simplest case: $n_i = 1$ and $m = n$, i.e. each agent knows exactly one row of \mathbf{A} matrix and one element of the b vector.

The distributed algorithm proposed in [12] computes the solution of the linear equation (1) through a multi-agent network $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The topology of this n -agent network is represented by its adjacency matrix $\mathcal{A}(\mathcal{G}) = [\alpha_{ij}]_{n \times n}$ with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

Agent i in the network is synonymous with vertex i in the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The topology of the multi-agent network is completely independent of the linear equation in (1).

For simplicity we make the following assumption:

Assumption 1: The graph \mathcal{G} is undirected and connected. Every vertex has a self loop and there are no multiple edges between two vertices.

Consider agent i who knows (A_i, b_i) . It calculates its local solution $x_i \in \mathbb{R}^n$ to $A_i x_i = b_i$ and exchanges the solution x_i with its neighbors, denoted as $\mathcal{N}_i = \{j \in \mathcal{V} | (i, j) \in \mathcal{E}\}$. In this work t is a discrete time variable and takes values in $\{0, 1, 2, \dots\}$. The exact (or global) solution to $\mathbf{A}x = b$ is obtained when all the local solutions x_i 's reach consensus through the following iteration procedure:

$$x_i(t+1) = x_i(t) - \frac{1}{d_i} \mathbf{P}_i \left(d_i x_i(t) - \sum_{j \in \mathcal{N}_i} x_j(t) \right), \quad (2)$$

where $\mathbf{P}_i = \mathbf{I} - A_i^T (A_i \cdot A_i^T)^{-1} A_i$ is the orthogonal projection on the kernel of A_i , $i = 1, \dots, n$, and $d_i = \sum_{j=1}^n \alpha_{ij}$ is the degree of agent i .

Let x^* be the true solution to (1) and it must satisfy $A_i x^* = b_i$ for $i = 1, \dots, n$. Define the error between $x_i(t)$ and x^* as

$$y_i(t) = x_i(t) - x^*, \quad (3)$$

which is in the kernel of A_i . In addition, note that $\mathbf{P}_i^2 = \mathbf{P}_i$ and $\mathbf{P}_i y_i(t) = y_i(t)$. Replacing $x_i(t+1)$ and $x_i(t)$ by $y_i(t+1)$ and $\mathbf{P}_i y_i(t)$ in (2), we get the *error updating equation*

$$y_i(t+1) = \frac{1}{d_i} \mathbf{P}_i \sum_{j \in \mathcal{N}_i} \mathbf{P}_j y_j(t), \quad (4)$$

for $i = 1, \dots, n$. These n equations can be rewritten in the following compact form

$$y(t) = (\mathbf{P}_{\text{diag}} [(\mathbf{D}^{-1} \mathbf{A}^T) \otimes \mathbf{I}] \mathbf{P}_{\text{diag}})^t y(0) = \mathbf{M}^t y(0), \quad (5)$$

where the matrix \mathbf{M} is called the *updating matrix* and $y(t) = \text{col}\{y_1(t), y_2(t), \dots, y_n(t)\}$. The matrix $\mathbf{P}_{\text{diag}} = \text{diag}\{\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n\} \in \mathbb{R}^{n^2 \times n^2}$ is a block diagonal matrix with $\mathbf{P}_i \in \mathbb{R}^{n \times n}$ and $\mathbf{D} = \text{diag}\{d_1, d_2, \dots, d_n\}$ is a diagonal matrix. The operator \otimes is the Kronecker product.

This algorithm has been proven to converge by using the mixed norm [14] [12, Chapter 4.3.1] of \mathbf{M} defined as

$$\|\mathbf{M}\|_{\text{mix}} = \|\mathbf{Q}\|_{\infty},$$

where $\mathbf{Q} = \{q_{ij}\}$, $q_{ij} = \frac{\alpha_{ij}}{d_i} \|\mathbf{P}_i \mathbf{P}_j\|$. Indeed, \mathbf{M}^t satisfies $\lim_{t \rightarrow \infty} \|\mathbf{M}^t\|_{\text{mix}} = 0$ if the undirected multi-agent network is connected [12]. Therefore $y = \mathbf{M}^t y(0) \rightarrow 0$ and thus $x_i \rightarrow x^*$ for all $i \in \mathcal{V}$.

Network properties play important roles in consensus problems. In particular, the second smallest eigenvalue $\lambda_2(\mathcal{L})$ of the graph laplacian bounds the convergence rate of consensus. Given the fact that projection matrices \mathbf{P}_i 's are used in constructing the updating matrix \mathbf{M} , it is not clear how the network topology \mathcal{A} impacts the convergence rate of this algorithm. Thus, in this work we approach the proof of convergence from a different angle.

III. IMPACTS OF NETWORK TOPOLOGY ON THE DISTRIBUTED ALGORITHM

A. Theoretical Analysis

In this section, we study how network topology impacts the performance of the network-based distributed algorithm. Before we state the main theorem, we introduce the following definitions.

Definition 3.1 (Walk): In a graph \mathcal{G} , a walk $w^l \in \mathcal{V}^{l+1}$ of length l is a sequence of vertices (v_0, v_1, \dots, v_l) with $\{v_{i-1}, v_i\} \in \mathcal{E}(\mathcal{G})$ for all $1 \leq i \leq l$ when $l \geq 1$. If $l = 0$, then w^0 is simply a vertex v_0 . Specifically, we denote a walk of length l starting at vertex v_0 and ending at vertex v_l as $w_{v_0 v_l}^l$.

Definition 3.2 ($f(w^l, \beta)$ Product of a Walk): Let w^l be a walk of length l . Let $\beta_{v_i} \in U$ be a value associated with vertex v_i . We can define a function of the walk w^l as

$$f(w^l, \beta) = \prod_{i=0}^{l-1} \beta_{v_i},$$

where β is indexed by the walk $w^l = (v_0, v_1, \dots, v_l)$ with values $\beta = (\beta_{v_0}, \beta_{v_1}, \dots, \beta_{v_l})$. The function $f(w^l, \beta) \in U$ is called the *product of walk w^l* . In this work U is either \mathbb{R} or $\mathbb{R}^{n \times n}$.

Definition 3.3 ($\mathbb{S}(l)$ and $\mathbb{S}^1(l)$ Spaces): In a graph \mathcal{G} , all the possible walks of length l form the $\mathbb{S}(l)$ Space. Denote a subspace of $\mathbb{S}(l)$ as $\mathbb{S}^1(l)$ if and only if

- the walk w^l starts from an arbitrary vertex v_0 and ends at v_l and visits all the vertices $v_i \in \mathcal{V}$ of \mathcal{G} ,
- there does not exist a vertex $v_j \in \mathcal{V}$ that divides w^l into two sub-walks, where one walk starts at v_0 and ends at v_j , the other one starts at v_j and ends at v_l , that both of them visit all the vertices $v_i \in \mathcal{V}$ of \mathcal{G} .

Note that the end vertex of the previous sub-walk and the starting vertex of the following sub-walk are repeated twice when dividing a walk. It is trivial that for w^l walks of length $l \leq n-1$, they can't be in the $\mathbb{S}^1(l)$ subspace.

Definition 3.4 (Order r): If a walk w^l can be divided into several walks $w^{l_1}, w^{l_2}, \dots, w^{l_r}$, where $l_i \geq 1$ and $w^{l_i} \in \mathbb{S}^1(l_i)$, then all the walks of the same number r form a subspace $\mathbb{S}^r(l)$ where r is called the *order* of the space. We also say that r is the *order* of the walk w^l . $\mathbb{S}^r(l) \subsetneq \mathbb{S}(l)$ for any order r .

If a walk w^l does not visit all the vertices in a graph \mathcal{G} , then its order is $r = 0$ and it is in $\mathbb{S}^0(l)$. This special case means that there exists at least one vertex $v_i \in \mathcal{V}$ which does not appear in the sequence of the walk w^l . The order of any w^l walk is uniquely determined and non-negative, i.e. $r \geq 0$.

Let $\varphi = \frac{1}{(\sqrt{n} \tau \|\mathbf{A}^{-1}\|)^2}$, $\tau = \max_i (\|A_i\|)$, $\frac{1}{d} = \left(\frac{1}{d_i}, \frac{1}{d_{v_1}}, \dots, \frac{1}{d_j}\right)$ be indexed by the walk $w_{ij}^t = (i, v_1, \dots, v_{t-1}, j)$ which starts at agent i and ends at agent j where $w_{ij}^t \in \mathcal{V}^{t+1}$, then we have the following theorem

Theorem 3.5 (Convergence Bound): Given a linear equation $\mathbf{A}x = b$, $\mathbf{A} = \text{col}\{A_i\} \in \mathbb{R}^{n \times n}$ and its unique solution x^* , let $x_i(t)$ be the local solution at agent i located in an undirected network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ whose adjacency matrix is

$A = \{\alpha_{ij}\}$, then the error $y_i(t)$ defined in (3) is bounded as

$$\|y_i(t+1)\| \leq \sum_{\mathcal{N}_j} \sum_{r=0}^{r_m(t)} \sum_{w_{ij}^t \in \mathbb{S}^r} f(w_{ij}^t, \frac{1}{d}) (1-\varphi)^{\frac{nr}{2}} \|y_j(0)\| \quad (6)$$

for $i = 1, \dots, n$. Here $r_m(t) \leq \lfloor \frac{t}{n} \rfloor$ is the maximum order of the product. Note that $w_{ij}^0 = w_{ii}^0 = (i)$ and $w_{ij}^1 = (i, j)$.

Theorem 3.5 provides another method to prove that the distributed algorithm converges to the true solution x^* besides the mixed norm method in [12], which is discussed at the end of this work. The bound in (6) connects the network topology with the convergence rate of the algorithm, by the degree d_i of agent i explicitly, and by counting the number of $w^t \in \mathbb{S}^r(t)$ walks in every order $r \geq 0$ in the network implicitly. Before moving to the detailed proof of this theorem, we first discuss how topology impacts the performance of the algorithm. To illustrate the topology impacts, we start with the definition of a walk w^t , then we discuss the properties of the corresponding $f(w^t, \frac{1}{d})$ product.

Given a network \mathcal{G} of size n , all the possible walks of length t are determined by its adjacency matrix $\mathcal{A} = \{\alpha_{ij}\}$. Let $\frac{1}{d_i}$ be the inverse degree of agent i , then the product $\frac{1}{d_{i_0}} \frac{1}{d_{i_1}} \dots \frac{1}{d_{i_t}}$ can be represented by $f^r(w_{i_0 i_t}^t, \frac{1}{d})$, where we recall that $\frac{1}{d}$ is indexed by the walk $w_{i_0 i_t}^t$. For simplicity, we let $i = i_0$ and $j = i_t$. Hence given a starting agent i , the summation of all products of the walk w^1 from i to all the agents $j = 1, 2, \dots, n$ is represented as $\sum_{j=1}^n \frac{\alpha_{ij}}{d_i d_j}$. In general, we have

$$\sum_{r=0}^{r_m(t)} \sum_{w_{ij}^t} f^r(w_{ij}^t, \frac{1}{d}) = \sum_{l_{t-1}=1}^n \dots \sum_{l_1=1}^n \frac{\alpha_{i l_1} \alpha_{l_1 l_2} \dots \alpha_{l_{t-1} j}}{d_i d_{l_1} \dots d_{l_{t-1}} d_j}$$

It is trivial that for any r, i, j and the walk w_{ij}^t , $f(w_{ij}^t, \frac{1}{d}) \in (0, 1)$. We now explore a scenario when the above mentioned sum remains a constant, even if the walk length increases.

Given a network \mathcal{G} and given a starting agent i , if all walks w_{ij}^t , $j = 1, 2, \dots, n$ are repeated by walks $w_{ij'}^{t+1}$ who visit one more agent j' at the end, after reaching agent j , then the summation of all $f(w_{ij'}^{t+1}, \frac{1}{d})$ products remains the same. This visit of agent j' generates n products based on each $f(w_{ij}^t, \frac{1}{d})$ and each of them equals to $\frac{\alpha_{jj'}}{d_{j'}} f(w_{ij}^t, \frac{1}{d})$, $j = 1, 2, \dots, n$. Only d_j out of n products are not zero when $\alpha_{jj'} = 1$. The summation of all newly generated products is unchanged, which is

$$\sum_{\mathcal{N}_{j'}} \frac{1}{d_{j'}} \sum_{\mathcal{N}_j} f(w_{ij}^t, \frac{1}{d}) = \sum_{\mathcal{N}_j} f(w_{ij}^t, \frac{1}{d}) \quad (7)$$

for $\sum_{\mathcal{N}_{j'}} = d_{j'}$. In general, the summation of all products of all walks by $t+1$ visits starting from a given agent i to all the neighbors of all the agents j is

$$\begin{aligned} & \sum_{\mathcal{N}_j} \sum_{r=0}^{r_m(t)} \sum_{w_{ij}^t \in \mathbb{S}^r(t)} f(w_{ij}^t, \frac{1}{d}) \\ &= \sum_{j=1}^n \sum_j \sum_{i_{t-1}}^n \dots \sum_{i_1}^n \frac{\alpha_{i i_1}}{d_i} \dots \frac{\alpha_{i_{t-1} j}}{d_{i_{t-1}}} \frac{\alpha_{j j'}}{d_{j'}} = 1. \end{aligned} \quad (8)$$

Given a network \mathcal{G} and a starting agent i , the summation $\sum_{\mathcal{N}_j} \sum_{w_{ij}^t \in \mathbb{S}^0(t)} f(w_{ij}^t, \frac{1}{d})$ is never increasing and the order r of the $f(w^t, \frac{1}{d})$ product is never decreasing as the walk length t grows. Given an arbitrary $f(w_{i_0 i_t}^t, \frac{1}{d})$ product of the walk $w_{i_0 i_t}^t \in \mathbb{S}^0(t)$, when the walk $w_{i_0 i_t}^t$ makes one more visit from agent i_t to the next agent i_{t+1} , it forms d_{i_t} new products and the summation of all d_{i_t} products is unchanged, which is already shown in (7). However, there exists a walk of length t_1 when there exists at least one walk changing from the $\mathbb{S}^0(t_1)$ subspace to the $\mathbb{S}^0(t_1+1)$ subspace. For every w^{t_2} walk (of order $r \geq 1$) of length t_2 , it never changes to a walk of order $r = 0$. This hold for any walk $w^t \in \mathbb{S}^0(t)$, hence the summation of all $f(w^t, \frac{1}{d})$, $w^t \in \mathbb{S}^0(t)$ product is never increasing, that is

$$\sum_{\mathcal{N}_j} \sum_{w_{ij}^{t+1} \in \mathbb{S}^0(t+1)} f(w_{ij}^{t+1}, \frac{1}{d}) \leq \sum_{\mathcal{N}_j} \sum_{w_{ij}^t \in \mathbb{S}^0(t)} f(w_{ij}^t, \frac{1}{d})$$

and given a walk of length t and a starting agent i , the bound in (6) decreases when the order of walks increases, due to the exponential factor $\lim_{r \rightarrow \infty} (1-\varphi)^{\frac{nr}{2}} = 0$. Since the summation of all f products starting from a chosen agent i is always 1 (8), the bound in 3.5 can only be decreased by either i) for a fixed length t , increasing the percentage of walks with higher r , or ii) by increasing the order r for all walks as rapidly as possible.

With the above two observations we conclude that given any two networks \mathcal{G}_1 and \mathcal{G}_2 , the distributed algorithm (2) tends to converge faster on networks \mathcal{G}_1 if \mathcal{G}_1 and \mathcal{G}_2 have similar topology properties except any combinations of the following

- 1 \mathcal{G}_1 has a shorter diameter,
- 2 \mathcal{G}_1 has a more homogeneous degree distribution,
- 3 \mathcal{G}_1 has a higher mean degree.

Although Theorem 3.5 has $\frac{1}{d}$ as a factor in the products, it is not trivial to conclude that higher degree makes the products smaller since higher degree decreases each product while increases the number of products. The summation of all products remains a constant, as shown in (8). However the bound decreases when the order r of the products increases. We address these three points in order.

1) *Diameter*: For two graphs \mathcal{G}_1 and \mathcal{G}_2 with the same degree distribution and hence the same mean degree, if \mathcal{G}_1 has a shorter diameter than \mathcal{G}_2 , then for fixed t , walks from \mathcal{G}_1 will necessarily have a larger minimum order r as compared to those from \mathcal{G}_2 . This follows from the fact that all the agents can be visited with fewer steps in a network with shorter diameter. Thus, all things being equal between two graphs, if $r(t)$ increases more rapidly for one graph as opposed to another, the exponential factor $(1-\varphi)^{\frac{nr}{2}}$ will decrease more rapidly. Therefore networks with shorter diameter make the distributed algorithm converge faster.

2) *Degree Distribution*: Let \mathcal{G}_1 and \mathcal{G}_2 be two graphs with same mean degree but different degree distributions. Let \mathcal{G}_1 have a more homogeneous degree distribution than \mathcal{G}_2 . Walks in \mathcal{G}_2 typically have lower order r than the walks of the same length in \mathcal{G}_1 . This is because walks on \mathcal{G}_2 rather than

\mathcal{G}_1 have to walk through the high degree vertices again and again to reach all the other low degree vertices. Hence for a given length of walks, the order r from the walks on \mathcal{G}_1 is higher. Therefore homogeneous degree distribution makes the algorithm converges faster.

3) *Mean Degree*: Adding edges to a graph typically results in a shorter diameter. Given two graphs \mathcal{G}_1 and \mathcal{G}_2 with similar degree distribution where \mathcal{G}_1 has a higher mean degree, the diameter of \mathcal{G}_1 is typically no larger than \mathcal{G}_2 . Hence the orders r 's from \mathcal{G}_1 are typically higher than those in \mathcal{G}_2 for walks of fixed length. Adding a new edge can either make the degree distribution homogeneous or make it heterogeneous, depending on where the new edge is added. The overall change of degree distribution for each newly added edge is difficult to analyze. However, if multiple new edges are added uniformly to a graph, this will typically result in a more homogeneous degree distribution, thus increasing the mean degree of the network makes the distributed algorithm converge faster.

B. Simulation Results

To verify our theoretical predictions, we perform extensive numerical simulations. We first quantify the convergence rate of the network-based distributed algorithm. One measure is the solution accuracy of the algorithm, which is the Euclidean distance between the local solution and the exact (or global) one:

$$\epsilon_i(t) = \|x_i(t) - x^*\|, \quad i = 1, 2, \dots, n.$$

Smaller ϵ_i means faster convergence rate and hence better algorithm performance. The impacts of different network topologies are measured by the statistical performances of the distributed algorithm, i.e. $E(\sum_{i=1}^n \epsilon_i)$ on an ensemble of linear equations. We notice that the Euclidean distance defined above needs a reference. For example, if the true solutions of two cases are $\|x^{*,1}\| = 100$ and $\|x^{*,2}\| = 0.1$ respectively, while the summation of Euclidean distances of all local solutions to $x^{*,j}$ are both $\sum_{i=1}^n \epsilon_i^j = \sum_{i=1}^n \|x_i^j - x^{*,j}\| = 1$, $j = 1, 2$, it is obvious the accuracy of the former iterative process is much higher than the latter one. Therefore the Euclidean distance should be scaled by the initial error $\sum_{i=1}^n \epsilon_i(0)$, yielding the relative error

$$R(t) = \frac{\sum_{i=1}^n \epsilon_i(t)}{\sum_{i=1}^n \epsilon_i(0)} = \frac{\sum_{i=1}^n \|x_i(t) - x^*\|_2}{\sum_{i=1}^n \|x_i(0) - x^*\|_2}. \quad (9)$$

In this way, convergence performances among a system of linear equations can be compared.

Figure. 1 shows the relative error changes with different network topologies, including small-world (SW) networks [15] with random rewiring probability p , scale-free (SF) networks [16] with degree exponent γ , Erdős-Rényi (ER) random graphs [17] with connectivity probability p and random regular (RR) graphs [18] with mean degree $\langle k \rangle$. The networks in each subfigure are the same in their mean degree and they are different on only one parameter. Small-world networks (a-c) are different in rewiring probabilities p , which determines network diameters. Scale-free networks,

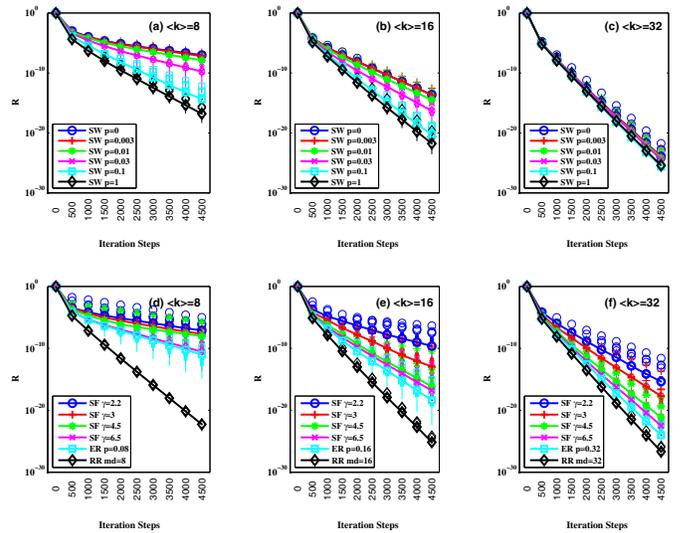


Fig. 1. Impact of network topology on the performance of the network-based distributed algorithm. Tens of different linear equations are solved by the distributed algorithm on six groups of networks of size $n = 100$. The complex networks in each group are (a-c) Small-world (SW) networks; (d-f) Scale-free (SF) networks, Erdős-Rényi (ER) random graphs, random regular (RR) graphs. In each case, we show the box-and-whisker plots and the median value of the relative error (or convergence rate) $R(t)$ as functions of t . At each marked iteration step t , a box-and-whisker plot is drawn. The mean degree of the complex networks is represented as $\langle k \rangle$.

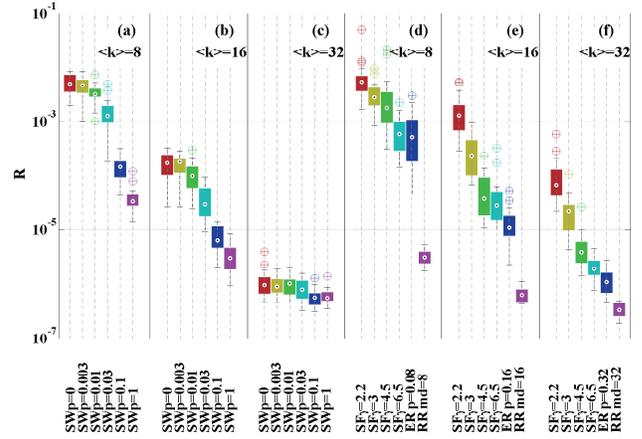


Fig. 2. Convergence rate at a chosen time step for complex networks with different topologies. The box-plot shows the relative errors at a given step $T_s = 2000$. Networks with similar topological features are grouped together in a particular subfigure.

graphs and RR graphs are drastically different in their degree distributions: scale-free networks are most heterogeneous and random regular graphs are most homogeneous.

The numerical results shown in Figure. 1 clearly verify our theoretical predictions, i.e. if two networks share similar topological properties, the one with smaller diameter (or more homogeneous degree distribution, or higher mean degree) perform better than the other. To further demonstrate the topology impacts, consider $R(t)$ at $t = 2000$ shown as box-and-whisker plots in Figure. 2. The smaller relative error $R(t)$ means higher convergence rate. It is clear from Figure. 2a-c and Figure. 2d-f that the upper bound of relative

errors decreases as the mean degree increases for a given network model. In other words, higher mean degree makes the algorithm reach the true solution faster, and is consistent with our theoretical analysis. Figure. 2a-c display that small-world networks with higher rewiring probability (and hence smaller diameters) have smaller relative errors R , confirming our theoretical prediction smaller diameter contributes to higher convergence rate. As shown in Figure. 2d-f, for any given mean degree, the random regular graphs have the smallest relative errors while scale free networks perform the worst. This means that the degree heterogeneity degrades the performance of the network-based distributed algorithm in solving linear equations (1).

IV. PROOF OF THE BOUND THEOREM

Before the formal proof of Theorem 3.5, we discuss the structure of the matrix \mathbf{M}^t (5) and introduce some technical lemmas.

Let $m_{ij}^{(1)} \in \mathbb{R}^{n \times n}$ be the i, j -th partition matrix of \mathbf{M} , then

$$m_{ij}^{(1)} = \frac{\alpha_{ij}}{d_i} \mathbf{P}_i \cdot \mathbf{P}_j,$$

where we recall that \mathbf{P}_i is an orthogonal projection matrix defined right after (2). These block matrices $m_{ij}^{(1)}$ are actually the updating matrix of $y_i(t)$, which means $y_i(t+1) = \sum_{j=1}^n m_{ij}^{(1)} y_j(t)$. Similarly, let $m_{ij}^{(t)}$ denote the partition matrix of \mathbf{M}^t , then

$$\begin{aligned} m_{ij}^{(t)} &= \sum_{l_{t-1}=1}^n \cdots \sum_{l_1=1}^n m_{il_{t-1}} \cdots m_{l_{t-1}l_1} \\ &= \sum_{l_{t-1}=1}^n \frac{\alpha_{l_{t-1}j}}{d_{l_{t-1}}} \cdots \sum_{l_1=1}^n \frac{\alpha_{il_1} \cdot \alpha_{l_1 l_2}}{d_i \cdot d_{l_1}} \mathbf{P}_i \cdots \mathbf{P}_{l_{t-1}} \mathbf{P}_j. \end{aligned}$$

Although the expression of $m_{ij}^{(t)}$ is long, it shows that \mathbf{M}^t is simply a weighted sum of projection products. It follows that (4) can be written as $y_i(t) = \sum_{j=1}^n m_{ij}^{(t)} y_j(0)$. Define $\mu_{ij} = \frac{\alpha_{ij}}{d_i} \in [0, 0.5]$, then we have

$$y_i(t) = \sum_{j=1}^n \cdots \sum_{l_1=1}^n \mu_{il_1} \cdots \mu_{l_{t-1}l_1} \mathbf{P}_i \cdots \mathbf{P}_{l_{t-1}} \mathbf{P}_j y_j(0). \quad (10)$$

Note that it is a summation of n^t products. We now separate $\mu_{il_1} \mu_{l_1 l_2} \cdots \mu_{l_{t-1}l_1} \mathbf{P}_i \mathbf{P}_{l_1} \cdots \mathbf{P}_{l_{t-1}} \mathbf{P}_j y_j(0)$ into a μ product

$$\mu_{il_1} \mu_{l_1 l_2} \cdots \mu_{l_{t-1}l_1} \quad (11)$$

and its corresponding projection product with $y_j(0)$, which is called *error sequence*,

$$\mathbf{P}_i \mathbf{P}_{l_1} \cdots \mathbf{P}_{l_{t-1}} \mathbf{P}_j y_j(0). \quad (12)$$

From (7) the summation of all μ products (11) satisfies the following equality

$$\sum_{j=1}^n \sum_{l_{t-1}=1}^n \cdots \sum_{l_1=1}^n \mu_{il_1} \mu_{l_1 l_2} \cdots \mu_{l_{t-1}l_1} = 1. \quad (13)$$

The construction of \mathbf{M}^t as a μ product and an error sequence of projections allows us to separate the topological

features from the part of the algorithm that is specific to a particular linear equation. We first analyse each product in the error updating equation (10) by bounding the error sequences of (12).

Define a sequence of vectors $z(t) \in \mathbb{R}^n$ as following

$$z^{(j)}(t+1) = z(t) + \frac{b_j - A_j z(t)}{\|A_j\|^2} A_j^\top, \quad (14)$$

where $t \geq 0$ and the superscript (j) corresponds to its row vector A_j and its scalar b_j . Then

$$\begin{aligned} \mathbf{P}_i(z(0) - x^*) &= z(0) - \frac{A_i z(0)}{\|A_i\|^2} A_i^\top - x^* + \frac{b_j}{\|A_i\|^2} A_i^\top \\ &= z(0) + \frac{b_i - A_i z(0)}{\|A_i\|^2} A_i^\top - x^* \\ &= z^{(i)}(1) - x^*. \end{aligned}$$

Let $z^{(j)}(0) = x_j(0)$, then each error sequence in (12) can be written as

$$\begin{aligned} &\mathbf{P}_i \mathbf{P}_{l_1} \cdots \mathbf{P}_{l_{t-2}} \mathbf{P}_{l_{t-1}} \mathbf{P}_j y_j(0) \\ &= \mathbf{P}_i \mathbf{P}_{l_1} \cdots \mathbf{P}_{l_{t-2}} \mathbf{P}_{l_{t-1}} \left(z^{(j)}(0) - x^* \right) \\ &= \mathbf{P}_i \cdots \mathbf{P}_{l_{t-2}} \left(z^{(j)}(0) + \frac{b_{l_{t-1}} - A_{l_{t-1}} z^{(j)}(0)}{\|A_{l_{t-1}}\|^2} A_{l_{t-1}}^\top - x^* \right) \\ &= \mathbf{P}_i \cdots \mathbf{P}_{l_{t-2}} \left(z^{(j_{l_{t-1}})}(1) - x^* \right) \\ &= z^{(i l_1 \cdots l_{t-2} l_{t-1} j)}(t) - x^*. \end{aligned} \quad (15)$$

Essentially, $z^{(i l_1 \cdots l_{t-2} l_{t-1} j)}(t)$ forms the sequence of $z(t)$ by taking different combinations of orthogonal projection \mathbf{P}_i at different agents, $i = 1, 2, \dots, n$. We now show that sequences $z(t)$ can be bounded, so that the error sequence is bounded as well.

We now present two theorems for bounding $z(t) - x^*$, first for the case when the walk w^t is associated with the product $f(w^t, \mathbf{P}_i)$, $w^t \in \mathbb{S}^0(t)$, and second for the $f(w^t, \mathbf{P}_i)$ product where $w^t \in \mathbb{S}^r(t)$ and $r \geq 1$.

Theorem 4.1 (f^0 Bound): For any $w^t \in \mathbb{S}^0(t)$ it follows that $\|f(w^t, \mathbf{P})\| \leq 1$ and thus $\|f(w^t, \mathbf{P})\| \leq 1$. Therefore the dynamics in (14) satisfy the following inequality

$$\|z(t) - x^*\| \leq \|z(0) - x^*\| \quad (16)$$

Proof: Given that \mathbf{P}_i is a normalized projection matrix it follows that $\|\mathbf{P}_i\| = 1$. \blacksquare

Theorem 4.2 (f Bound): The sequence $z(t) - x^*$ of the part whose $\mathbf{P}_i \mathbf{P}_{l_1} \cdots \mathbf{P}_{l_{t-1}} \mathbf{P}_j$ product is an $f(w^t, \mathbf{P})$ product where $w^t \in \mathbb{S}^r(t)$ and $r \geq 1$, then all the sequence $\mathbf{P}_i \mathbf{P}_{l_2} \cdots y_j(0)$ in this part from (15) can be written as

$$z(t) - x^* = f(w^t, \mathbf{P}) y_j(0),$$

where $z(t) - x^*$ consists of several $f(w^t, \mathbf{P})$, $w^t \in \mathbb{S}^1(i)$ products. Then all the sequences $z(t) - x^*$ in this part are bounded by

$$\begin{aligned} \|z(t) - x^*\| &\leq \left(1 - \frac{1}{(\sqrt{n\tau} \|\mathbf{A}^{-1}\|)^2} \right)^{\frac{nr}{2}} \|z(0) - x^*\| \\ &< (1 - \kappa(\mathbf{A})^{-2})^{\frac{nr}{2}} \|z(0) - x^*\|, \end{aligned}$$

where $\kappa(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$ is the usual condition number of \mathbf{A} and we recall the definition $\tau = \max_i (\|A_i\|)$.

The proof of Theorem 4.2 requires several technical Lemmas.

Lemma 1 (Orthogonal Projection): Let $z(t) \in \mathbb{R}^n$, $\|z(0)\| = 0$ be a sequence that follows

$$z^{(j)}(t+1) = z(t) + \frac{b_j - A_j z(t)}{\|A_j\|^2} A_j^T,$$

where A_j, b_j are defined as those in linear equation (1), which is the same as (14). Then the orthogonal projection matrix \mathbf{P}_i^* onto the solution space of the linear equation (1) is given in [19] as

$$z(t+1) = \mathbf{P}_i^* z(t).$$

Let $\langle z(t+1), z(t) \rangle$ denotes the inner product of two vectors $z(t+1)$ and $z(t)$, then the above equation can be written as follows by using the updating function (14)

$$\begin{aligned} \mathbf{P}_i^* z(t) &= z(t) - \frac{A_i z(t) - b_i}{\|A_i\|^2} A_i^T \\ &= z(t) - \frac{A_i z(t) - A_i z^*}{\|A_i\|} \frac{A_i^T}{\|A_i\|} \\ &= z(t) - \langle z(t) - z^*, Z_i \rangle Z_i^T, \end{aligned}$$

where $Z_i = \frac{A_i}{\|A_i\|}$, $i = 1, 2, \dots, n$, $\|Z_i\| = 1$ is a set of normal vectors in the hyperplane $\{z(t) : \langle A_i, z(t) \rangle = b_i\}$.

Lemma 2 (Orthogonality): Consider the linear equation (1) and let x^* be the unique solution. The difference of two vectors $z(t+1)$ and $z(t)$ is in the kernel of \mathbf{P}_i^* by Orthogonal Projection Lemma 1, which means that it is orthogonal to the solution space. Therefore it is also orthogonal to $z(t+1) - x^*$. In other words, the orthogonality of two vectors $z(t+1) - z(t)$ and $z(t+1) - x^*$ satisfies

$$\|z(t+1) - z(t)\|^2 + \|z(t+1) - x^*\|^2 = \|z(t) - x^*\|^2.$$

Lemma 3 (Inequality): Let $\mathbf{A} = \text{col}\{A_i\}$, $\mathbf{A} \in \mathbb{R}^{n \times n}$ is full rank. Then the following inequality holds

$$\sum_{i=1}^n \left\| \left\langle \frac{A_i}{\|A_i\|}, x \right\rangle \right\|^2 \geq \frac{1}{(\tau \|\mathbf{A}^{-1}\|)^2} \|x\|^2.$$

where $\langle A_i, x \rangle$ denotes the inner product of vector A_i and x and we recall the definition $\tau = \max_i (\|A_i\|)$.

Due to the page limitation, we skip the detailed proof of Lemma 3, Theorem 4.2, Theorem 3.5, as well as the discussion of the algorithm convergence. See Ref. [1] for all the details.

V. CONCLUSIONS

In this work, we studied the impact of network topology on the performance of a network-based distributed algorithm that solves linear algebraic equations. Both theoretical analysis and simulation results show that networks with higher mean degree, smaller diameter, and more homogeneous degree distribution make the algorithm converge faster. Interestingly, k -regular random networks with small mean degree can have comparable performance to a degree-heterogeneous

network with very high mean degree. Hence, it is possible to reduce the communication cost (i.e. by designing sparser networks) while maintaining the same convergence rate.

Besides classical consensus problems, we expect that more complicated problems can also be solved with network-based distributed algorithms. Our results presented here provide a method to analyze the topology impacts on a network-based distributed algorithm. It may shed light on the design of better network topologies to improve the performance of general multi-agent distributed algorithms in solving more challenging real-world problems.

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