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Adaptive Systems with Closed–loop Reference Models, Part I: Transient Performance

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***Abstract*—This paper explores the transient properties of direct model reference adaptive control with closed loop reference models. The transients are characterized by bounds on the model following error, the rate of change of the adaptive parameter and rate of change of the control input. Both Euclidean and $\mathcal{L}_2$ norms are used to characterize this transient behavior.**

I. Introduction

Early developments of adaptive systems included explorations of various kinds of reference models. The overall goal behind the selection of a reference model is that the corresponding *tracking error* must asymptotically decay in the absence of parametric uncertainties in the plant being controlled. In order to accomplish this goal, modifications of the *Open–loop Reference Models* (ORM) were explored [1], [2]. Some of these modifications retained stability properties and were otherwise indistinguishable from ORM–adaptive systems and as a result, not pursued. Others could not be shown to be stable and were therefore dropped. Recently, a class of *Closed–loop Reference Models* (CRM) have been proposed for control of plants with unknown parameters whose states are accessible (see for example [3]–[7]) all of which are guaranteed to be stable and additionally portray improved transient performance.

Transient behavior in adaptive systems have been addressed in recent years in [4], [6]–[8] and earlier in [9]. The results in [4] discussed the tracking error, but focused the attention mainly on the initial interval where the CRM-adaptive system exhibits fast time-scales. Results in [6], [7] focus on deriving a damping ratio and natural frequency for adaptive systems with CRM. However, assumptions are made that the initial state error is zero and that the closed-loop system state is independent of the feedback gain in the reference model, both of which may not hold in general. The results in [8] too assume that initial state errors are zero. And in addition, the bounds derived in [8] are based upon $\mathcal{L}_\infty$ norms, which do not capture the transient properties of adaptive systems. The results in [9] pertain to transient properties of adaptive systems, and quantify them using an $\mathcal{L}_2$ norm. The adaptive systems in question however are indirect, and do not pertain to CRMs.

In this paper, we propose CRM-based adaptive systems similar to [4], [6], [7]. Similar to these papers, we demonstrate their stability property and proceed to address their transient behavior. Unlike those papers, and similar to [9], our transient metric of choice is an $\mathcal{L}_2$–norm. We analyze the tracking error using both this norm and the Euclidean norm and show its improvement compared to ORM-adaptive systems. We also analyze derivatives of key signals including the adaptive parameter and the adaptive control input. We show that their Euclidean norm as well as their $\mathcal{L}_2$–norm are smaller than their ORM-counterparts.

We also establish yet another important feature of the CRM–adaptive systems, which is a water-bed effect. While the CRM-adaptive systems are shown to result in improved tracking errors and derivatives of key signals, they can introduce slow adaptation in the plant state and therefore a larger tracking error of the original reference model. Guidelines for an optimal CRM–design which ensures satisfactory transients are provided.

In comparison to [10], the previous work by the authors of this paper on the same topic, this paper consists of (i) new metrics of transients such as the $\mathcal{L}_2$ norm, and (ii) comparisons with the classical ORM-based MRAC. The same notations introduced in [10] are however used. A second part of this paper can be found in [11] which addresses CRM in composite MRAC and adaptive systems with observer–based feedback.

The results in this paper are organized as follows: Section II introduces the basic structure of CRM adaptive control as well as the Projection Operator. Section III investigates the transient response of CRM. Section IV compares ORM and CRM adaptive systems in terms of our performance metric. Section V contains our concluding remarks.

II. The CRM–Adaptive System

In this section, we describe the CRM–adaptive system, and establish its stability and convergence properties in the absence of any perturbations other than parametric uncertainties. We first describe the CRM–adaptive system and prove its closed–loop stability. After some preliminaries on matrix bounds, we introduce a projection algorithm in the adaptive law. This is used to derive exponentially converging bounds on the key variables in the CRM–adaptive system.

Consider the linear system dynamics with scalar input

$$\dot{x}(t) = A_p x(t) + bu(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $A_p \in \mathbb{R}^{n\times n}$ is unknown and $b \in \mathbb{R}^n$ is known. Our goal is to design the control input such that $x(t)$ follows the reference model state $x_m(t) \in \mathbb{R}^n$ defined by the following

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closed–loop dynamics

$$\dot{x}_m(t) = A_m x_m(t) + br(t) - L(x(t) - x_m(t)) \quad (2)$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwtitz and $r(t) \in \mathbb{R}$ is a bounded possible time varying reference command. $L \in \mathbb{R}^{n\times n}$ is denoted as the *Luenberger–gain*, and is chosen such that

$$\bar{A}_m \triangleq A_m + L \quad (3)$$

is Hurwitz. When $L = 0$ the classical ORM is recovered.

Assumption 1. A parameter vector $\theta^* \in \mathbb{R}^n$ exists that satisfies the *matching condition*

$$A_m = A_p + b\theta^{*T}. \quad (4)$$

The control input is chosen in the form

$$u(t) = \theta^T(t)x(t) + r(t) \quad (5)$$

where $\theta(t) \in \mathbb{R}^n$ is the adaptive control gain with the update law

$$\dot{\theta}(t) = -\Gamma x(t)e(t)^T Pb \quad (6)$$

with $\Gamma = \Gamma^T > 0$, $e(t) = x(t) - x_m(t)$ is the model following error and $P = P^T > 0$ is the solution to the algebraic Lyapunov equation

$$\bar{A}_m^T P + P\bar{A}_m = -I_{n\times n}. \quad (7)$$

The underlying error model in this case is given by

$$\dot{e}(t) = \bar{A}_m e(t) + b\tilde{\theta}(t)x(t) \quad (8)$$

where $\tilde{\theta}(t) = \theta(t) - \theta^*$ is the parameter error.

Theorem 1. *The closed-loop adaptive system with* (1), (2), (5) *and* (6) *is globally stable with* $e(t)$ *tending to zero asymptotically, under the matching condition in* (4).

Proof: It is straight forward to show using (6) and (8) that

$$V(e,\tilde{\theta}) = e^T Pe + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \quad (9)$$

is a Lyapunov function. Since e is bounded, the structure of (2) implies that x_m is bounded. x in turn and u are bounded. Barbalat lemma ensures asymptotic convergence of $e(t)$ to zero. □

Corollary 2. *For all* $\epsilon > 0$ *there exists* $T(\epsilon, L) > 0$ *such that* $t \geq T(\epsilon, L)$ *implies* $\|e(t)\| \leq \epsilon$.

The overall CRM–adaptive system is defined by (1), (2), (5), and (6). The standard open-loop reference model is given by

$$\dot{x}_m^o(t) = A_m x_m^o(t) + br(t) \quad (10)$$

with the corresponding tracking error

$$e^o(t) = x(t) - x_m^o(t). \quad (11)$$

One can in fact view the error e^o as the *true tracking error* and e as a *pseudo–tracking error*. The question that arises is whether the convergence properties that are assured in an ORM–adaptive system, of $e^o(t)$ tending to zero, is guaranteed in a CRM–adaptive system as well. This is addressed in the following corollary:

Corollary 3. *The state vector* $x(t)$ *converges to* $x_m^o(t)$ *as* $t \to \infty$.

Proof: From Theorem 1 we can conclude that $e(t) \to 0$ asymptotically. Thus we can conclude that $x_m(t) \to x_m^o(t)$ as $e(t) \to 0$, emplying that $e^o \to 0$, thus $x(t) \to x_m^o(t)$ as $t \to \infty$. □

Remark 1. The choice of the CRM as in (2) essentially makes the reference model nonlinear, as x depends on θ which in turn depends on x_m in a highly nonlinear manner. As we will show in Section III, the CRM-adaptive system has an additional desirable property, of quantifiable transient properties. We will also show in this section that this is made possible by virtue of the additional degree of freedom available to the adaptive system in the form of the feedback gain in the CRM.

A. Preliminaries

All norms unless otherwise noted are the Euclidean norm and the induced Euclidean norm. The other norms used in this work are the $\mathcal{L}_2$ and the $\mathcal{L}_\infty$ norm defined below. Given a vector $\nu \in \mathbb{R}^n$ and finite $p \in \mathbb{N}_{>0}$ $\|\nu(t)\|_{L_p} \triangleq \left(\int_0^\infty \|\nu(s)\|^p ds\right)^{1/p}$ and $\|\nu(t)\|_{L_p,\tau} \triangleq \left(\int_0^\tau \|\nu(s)\|^p ds\right)^{1/p}$. The infinity norm is defined as $\|\nu(t)\|_{L_\infty} \triangleq \sup\|\nu(t)\|$.

Definition 1. Given a Hurwtiz matrix $A_m \in \mathbb{R}^{n\times n}$

$$\begin{aligned}\sigma &\triangleq -\max_i\left(\text{real}(\lambda_i(A_m))\right)\\ s &\triangleq -\min_i\left(\lambda_i\left(A_m + A_m^T\right)/2\right)\\ a &\triangleq \|A_m\|.\end{aligned} \quad (12)$$

For ease of exposition, throughout the paper, we choose L in (2) and Γ in (17) and (43) as follows:

$$L \triangleq -\ell I_{n\times n} \quad (13)$$

$$\Gamma \triangleq \gamma I_{n\times n}. \quad (14)$$

Lemma 4. *The constants* σ *and* s *are strictly positive and satisfy* $s \geq \sigma > 0$.

Lemma 5. *With* L *chosen as in* (13), A_m *Hurwitz with constants* σ *and* s *as defined in* (12), P *in* (7) *satisfies*

$$\text{(i)} \qquad \|P\| \leq \frac{m^2}{\sigma + 2\ell} \quad (15)$$

$$\text{(ii)} \qquad \min_i \lambda_i(P) \geq \frac{1}{2(s+\ell)} \quad (16)$$

where $m = (1 + 4\varkappa)^{n-1}$ *and* $\varkappa \triangleq \frac{a}{\sigma}$.

Proof: See [10, Lemma 2].

B. Projection Algorithm

Before we evaluate the benefits of closed–loop reference models, we introduce a modification in the adaptive law to ensure robustness properties.

Assumption 2. A known θ^*_{max} exists such that $\|\theta^*\| \leq \theta^*_{max}$.

The projection based adaptive law, which replaces (6), is given by

$$\dot{\theta}(t) = \text{Proj}_\Gamma\left(\theta(t), -xe^T Pb, f\right) \tag{17}$$

where the Γ–projection function, Proj_Γ, is defined as in Appendix A and f is a convex function given by

$$f(\theta; \vartheta, \varepsilon) = \frac{\|\theta\|^2 - \vartheta^2}{2\varepsilon\vartheta - \varepsilon^2} \tag{18}$$

where ϑ and ε are positive constants chosen as $\vartheta = \theta^*_{\max}$ and $\varepsilon > 0$.

Definition 2. Using the design parameters of the convex function $f(\theta; \vartheta, \varepsilon)$ we introduce the following definitions

$$\begin{aligned} &\theta_{\max} \triangleq \vartheta + \varepsilon \text{ and} \\ &\tilde{\theta}_{\max} \triangleq 2\vartheta + \varepsilon. \end{aligned} \tag{19}$$

C. Convergence of the Adaptive System

Theorem 6. *Let Assumptions 1 and 2 hold. Consider the adaptive system defined by the plant in* (1) *with the reference model in* (2)*, the controller in* (5)*, the adaptive tuning law in* (17) *and L and Γ as in* (13)-(14)*. For any initial condition in $e(0) \in \mathbb{R}^n$, and $\theta(0)$ such that $\|\theta(0)\| \leq \theta_{max}$, $e(t)$ and $\theta(t)$ are uniformly bounded for all $t \geq 0$ and the trajectories in the Lyapunov candidate in* (9) *converges exponentially to a set $\mathcal{E}$ as*

$$\dot{V} \leq -\alpha_1 V + \alpha_2 \tag{20}$$

where

$$\alpha_1 \triangleq \frac{\sigma + 2\ell}{m^2} \text{ and } \alpha_2 \triangleq \frac{\sigma + 2\ell}{m^2\gamma}\tilde{\theta}^2_{max}, \tag{21}$$

and $\mathcal{E} \triangleq \left\{(e, \tilde{\theta}) \,\middle|\, \|e\|^2 \leq \beta_1 \tilde{\theta}^2_{max},\ \|\tilde{\theta}\| \leq \tilde{\theta}_{max}\right\}$ *with*

$$\beta_1 = 2\frac{s + \ell}{\gamma}. \tag{22}$$

Proof: See Appendix C.

III. Transient Performance of CRM–adaptive systems

In the following subsections we derive the transient properties of the CRM-adaptive systems. Five different subsections are presented, the first of which quantifies the Euclidean and the $\mathcal{L}_2$–norm of the tracking error e. In the second subsection we compute the same norms for the parameter derivative $\dot{\theta}(t)$. In both cases, we show that the $\mathcal{L}_2$–norms can be decreased by increasing ℓ. In the third theorem, we address the performance of the true error e^o and show its dependence on ℓ. In the fourth subsection, we define our metric for transient performance in terms of a truncated $\mathcal{L}_2$ norm of the rate of control effort. The last subsection compares ORM and CRM adaptive systems using these metrics.

Let

$$\rho = \frac{\gamma}{\sigma + \ell}. \tag{23}$$

The results in the following subsections are presented in terms of the two free design parameters ρ and ℓ, which is just a reparameterization of γ and ℓ. Then it is assumed that ρ is chosen independent of ℓ so that the product ΓP is of the same size while ℓ is being adjusted, where we note that

$$\|\Gamma\|\|P\| \leq \rho m^2. \tag{24}$$

This follows from the bound given in (15).

A. Bound on $e(t)$

Theorem 7. *Let Assumptions 1 and 2 hold. Consider the adaptive system defined by the plant in* (1) *with the reference model in* (2)*, the controller in* (5)*, the adaptive tuning law in* (17) *and L and Γ as in* (13) *and* (14)*.*

$$\|e(t)\|^2 \leq \kappa_1 \|e(0)\|^2 \exp\left(-\frac{\sigma + 2\ell}{m^2} t\right) + \frac{\kappa_2}{\rho}\tilde{\theta}^2_{max} \tag{25}$$

$$\|e(t)\|^2_{L_2} \leq \frac{1}{\sigma + \ell}\left(m\|e(0)\|^2 + \frac{1}{\rho}\|\tilde{\theta}(0)\|^2\right) \tag{26}$$

where κ_i, $i = 1, 2, 3$ are independent of ρ and ℓ.

Proof: see Appendix D. □

B. Bound on $\dot{\theta}(t)$

In addition to $\|e(t)\|_{L_2}$ we explicitly compute upper bounds for $\|\dot{\theta}(t)\|$ and $\|\dot{\theta}(t)\|_{L_2}$. From the definition of $\dot{\theta}(t)$ in (17), it follows that

$$\|\dot{\theta}(t)\| \leq \|\Gamma\|\|P\|\|b\|\|x(t)\|\|e(t)\|.$$

We note that $x(t) = e(t) + x_m(t)$ and from (2) and (44) that

$$\begin{aligned} \|x_m(t)\| \leq & x_m(0) m \exp\left(-\tfrac{\sigma}{2} t\right) \\ & + m \int_0^t \exp\left(-\tfrac{\sigma}{2}(t - \tau)\right)\left(l\|e\| + \|b\|\|r\|\right) d\tau \end{aligned} \tag{27}$$

Using the bound for $\|e(t)\|_{L_2}$ from (17) and the Cauchy–Schwartz inequality, we simplify (27) as

$$\|x_m(t)\| \leq x_m(0) m \exp\left(-\tfrac{\sigma}{2} t\right) + \frac{lm}{\sqrt{\sigma}}\|e(t)\|_{L_2} + \frac{r_0 2\|b\|m}{\sigma}. \tag{28}$$

The above bounds make the following theorem possible.

Theorem 8. *Let Assumptions 1 and 2 hold. Consider the adaptive system defined by the plant in* (1) *with the reference model in* (2)*, the controller in* (5)*, the adaptive tuning law in* (17) *and L and Γ as in* (13) *and* (14)*.*

$$\begin{aligned} \|\dot{\theta}(t)\| \leq & \rho \exp\left(-\tfrac{\sigma+2\ell}{2m^2} t\right)\left[a_1 + \sqrt{\ell}\left(a_2 + a_3\sqrt{\tfrac{1}{\rho}}\right)\right] \\ & + \sqrt{\rho}\exp\left(-\tfrac{\sigma}{2} t\right) a_4 + \sqrt{\tfrac{1}{\rho}}\exp\left(-\tfrac{\sigma+2\ell}{m^2}\right) a_5 \\ & + \sqrt{\ell\rho}\, a_6 + \sqrt{\ell}\, a_7 + \rho a_8 \end{aligned} \tag{29}$$

$$\|\dot{\theta}(t)\|^2_{L_2} \leq \rho^2 \nu_0(\rho)\left(\frac{b_1}{\sqrt{\sigma + \ell}} + \sqrt{\nu(\rho)}\, b_2 + \frac{b_3}{\sqrt{\sigma + \ell}}\right)^2 \tag{30}$$

where $\nu(\rho) = m\|e(0)\|^2 + \frac{1}{\rho}\|\tilde{\theta}(0)\|^2$, and the a_i and b_i are independent of ρ and ℓ.

Proof: see Appendix E. □

C. Bound on $e^o(t)$

While $e(t)$ denotes the error between the CRM and the closed–loop system, the true error that is of interest is $e^o(t)$ defined in (10).

Theorem 9. *Let the assumptions from Theorem 8 hold. The difference between the open–loop reference model and the closed loop reference model satisfy the following bound*

$$\|e^o(t)\| \leq \|e(t)\| + \sqrt{\frac{\ell}{\sigma}} m \sqrt{\nu(\rho)}. \tag{31}$$

Proof: see Appendix F □

D. Bound on $\dot{u}(t)$

We now derive a final transient measure of the CRM–adaptive system that pertains to $\dot{u}$. This is chosen as the transient performance metric because the rate of change of the control authority requested by the controller directly affects the robustness of the system to unmodelled dynamics and actuator rate limits. Before the bounds are derived, several variables must be defined.

Definition 3. Let time-constants $\tau_1(\ell)$, τ_2 be defined as

$$\tau_1(\ell) = \frac{2m^2}{\sigma + 2\ell} \text{ and } \tau_2 = \frac{2}{\sigma} \tag{32}$$

Let constants a_θ and $\delta_1(\ell, N)$ be defined as

$$\begin{aligned} a_\theta \triangleq& a + \|b\|\tilde{\theta}_{\max}, \\ \delta_1(\ell, N) =& \exp\left(a_\theta N \tau_1(\ell)\right) - 1. \end{aligned} \tag{33}$$

where $N > 0$, and three intervals of time

$$\begin{aligned} \mathbb{T}_1 &= [0, N\tau_1) \\ \mathbb{T}_2 &= [N\tau_1, T_1) \\ \mathbb{T}_3 &= [T_1, \infty) \end{aligned} \tag{34}$$

where $T_1 \triangleq \max\{N\tau_2, T(\epsilon, -\ell I_{n\times n})\}$ and $T(\epsilon, -\ell I_{n\times n})$ is defined in Corollary 2.

Remark 2. $t_1(\ell)$ is a time constant associated with the exponential decay of $\|e(t)\|$ which is derived from the upper bound on V from (20) and τ_2 is the time constant associated with A_m in (2). a_θ is a positive scalar that upper bounds the open–loop eigen values of A_p from (2) and $\delta_1(\ell)$ will be used in the following Lemma to formally define our time scale separation condition. The time interval $\mathbb{T}_1$ is the time interval over which $\|e(t)\|$ decays by N time constants, $\mathbb{T}_3$ is the asymptotic time scale for $e(t)$ and $\mathbb{T}_2$ is an intermediate time interval. We note that T_1 exists but is unknown.

Lemma 10. *For any $N > 0$ an ℓ^* exists such that*

(i) $\delta_1(\ell^*, N) < \delta$ *where* $0 < \delta \leq 1$.

(ii) $\tau_1(\ell^*) \leq \tau_2$.

Remark 3. The condition Lemma 10 (i) defines the time scale separation condition. Recall that τ_1 is the time scale associated with $e(t)$ and a_θ is an upper bound on the uncertain open–loop eigen values of the plant. When $\ell \geq \ell^*$ we are able to show that at $t_N = N\tau_1$, $e(t_N)$ has exponentially decade by N time constants, while $x(t_N)$ has not deviated far from $x(0)$.

Assumption 3. $\exists r_0, r_1 > 0$ s.t. $|r(t)| \leq r_0$, $|\dot{r}(t)| \leq r_1$.

Remark 4. The bound on $\dot{r}(t)$ is needed so that $\dot{u}(t)$ is well defined. The analysis techniques that follow in proving Theorem 11 will still hold for reference inputs with discontinuities. The metric for transient performance however would change from $\dot{u}$ to $\frac{d}{dt}\left(\theta^T(t)x(t)\right)$.

Assumption 4. For ease of exposition we will assume that $x_m(0) = 0$. We note that the same analysis holds for $x_m(0)$ with addition of exponentially decaying terms proportional to $x_m(0)$.

Theorem 11. *Let Assumptions 1–4 hold. Given arbitrary initial conditions in $x(0) \in \mathbb{R}^n$ and $\|\theta(0)\| \leq \theta_{max}$, for any $\epsilon > 0$, $N > 0$ and $\ell \geq \ell^*$, $\dot{u}$ satisfies the following inequalities:*

$$\begin{aligned} \sup_{t\in\mathbb{T}_i} |\dot{u}(t)| \leq& \frac{m^2\gamma}{\sigma + 2\ell}\|b\| G_{e,i} G_{x,i}^2 \\ &+ \theta_{max}\left(a_\theta G_{x,i} + r_0\right) + r_1 \end{aligned} \tag{35}$$

for $i = 1, 2, 3$, where

$$\begin{aligned} G_{x,1} \triangleq& (1+\delta_1)\|e(0)\| + \frac{\delta_1\|b\|}{a_\theta} r_0 \\ G_{e,1} \triangleq& \sqrt{\kappa_1}\|e(0)\| + \sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{max} \\ G_{x,2} \triangleq& \kappa_3\|e(0)\| + (1+\kappa_4\ell)\sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{max} + \kappa_5 r_0 \\ G_{e,2} \triangleq& \sqrt{\kappa_1}\|e(0)\|\epsilon_1 + \sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{max} \\ G_{x,3} \triangleq& \kappa_6\|e(0)\| + \epsilon + (1+\kappa_4\ell)\sqrt{\frac{\kappa_2}{\rho}}\tilde{\theta}_{max} + \kappa_5 r_0 \\ G_{e,3} \triangleq& \epsilon \end{aligned} \tag{36}$$

where $\epsilon_1 \triangleq \exp(-N)$ and the κ_i are independent of ρ and ℓ, and $N \geq 3$

Proof: see Appendix G.

Remark 5. There are two "small" terms in the above analysis, ϵ and ϵ_1. ϵ_1 is determined by the number of time constants N of interest. ϵ is free to choose and from Corollary 2 proves the existence of a finite T and is used to define when $\mathbb{T}_3$ begins.

From Theorem 11, it follows that

$$\begin{aligned} \sup_{t\in\mathbb{T}_1} |\dot{u}(t)| \leq& c_1\rho + c_2\sqrt{\rho} + r_1 \\ \sup_{t\in\mathbb{T}_2} |\dot{u}(t)| \leq& \sqrt{\rho}c_3 + (1+c_4 l)c_5 + \sqrt{\frac{1}{\rho}}(1+c_4\ell)^2 c_6 \\ &+ \epsilon_1\mathfrak{L}_1(\rho, \ell, \sqrt{\rho}, \ell\sqrt{\rho}, \ell^2) + r_1 \\ \sup_{t\in\mathbb{T}_3} |\dot{u}(t)| \leq& \sqrt{\frac{1}{\rho}}(1+c_4\ell)c_7 + c_8 \\ &+ \epsilon\mathfrak{L}_2(\rho, \ell, \sqrt{\rho}, \ell\sqrt{\rho}, \ell^2, \epsilon_1) + r_1 \end{aligned} \tag{37}$$

where $c_i > 0$, $i = 1$ to 8 are independent of ℓ and ρ, $\mathfrak{L}_1(\cdot)$ and $\mathfrak{L}_2(\cdot)$ are globally lipschitz with respect to their arguments. The inequalities in (37) lead us to the following three main observations (see Figure 1)

(A1) Over $\mathbb{T}_1$, $|\dot{u}(t)|$ is bounded by a linear function of ρ and $\sqrt{\rho}$,

(A2) Over $\mathbb{T}_2$, $|\dot{u}(t)|$ is bounded by a linear function of $\sqrt{\rho}, \ell, \sqrt{\frac{1}{\rho}}, \ell\sqrt{\frac{1}{\rho}}$ and $\ell^2\sqrt{\frac{1}{\rho}}$

(A3) Over $\mathbb{T}_3$, $|\dot{u}(t)|$ is bounded by a linear function of $\sqrt{\frac{1}{\rho}}$ and $\ell\sqrt{\frac{1}{\rho}}$

(A4) τ_1 decreases with ℓ.

Remark 6. The main idea used for the derivation of the bounds in Theorem 11 is time–scale separation of the error dynamics decay, and the worst case open–loop eigen values of the uncertain plant. The most important point to note is that τ_1 can be made as small by choosing a large ℓ. There is a penalty, however, in choosing a large ℓ, as the bound $G_{x,2}$ increases linearly with ℓ. Therefore, after choosing an ℓ witch satisfies the time scale separation as needed in Lemma 10, a ρ (which through (23) defines a choice for γ) can be chosen such that the integral in the following theorem is minimized.

Theorem 12. *There exist optimal ρ and ℓ such that*

$$(\rho_{opt}, \ell_{opt}) = \underset{\substack{\rho>0\\ \ell\geq\ell^*}}{\arg\min}\|\dot{u}(\rho, \ell)\|_{L_2,\tau} \tag{38}$$

for any $0 < \tau < T_1$.

Proof: $\|\dot{u}(\rho, \ell)\|_{L_2,\tau}$ is continuous with respect to ρ and ℓ where ρ and ℓ appear in the numerator of (37) and are positive. Therefore, ρ_{opt} and ℓ_{opt} exist and are finite. □

τ in Theorem 12 denotes the interval of interest in the adaptive system where the transient response is to be contained. Given that T, and therefore T_1 is a function of ℓ, (38) can only be minimized over $\mathbb{T}_1 \cup \mathbb{T}_2$. From the authors definition of smooth transient performance in the beginning of this section choosing ρ_{opt} and ℓ_{opt} will guarantee smooth transient performance.

E. Comparison of CRM and ORM-adaptive systems

The bounds on $e(t)$ and the $\mathcal{L}_2$–norm of $\dot{\theta}$ directly show that CRM–adaptive systems lead to smaller $e(t)$ than with the ORM which are obtained by settling $\ell = 0$ in (25) and (26). However, the same cannot be said for either e^o or for the Euclidean norm of $\dot{\theta}$; for a non-zero ℓ, the bound on e^o is larger than that of e. This indicates that there is a trade-off between fast transients and true tracking error. The signal that succinctly captures this trade off is $\dot{u}$, whose behavior is captured in detail using the time intervals $\mathbb{T}1$, $\mathbb{T}2$, and $\mathbb{T}3$. We also showed in Theorem 12 that this trade-off can be optimized via a suitable choice of ℓ and ρ. In what follows, we compare this optimized CRM with ORM and show that the former is clearly better than the latter.

Definition 4. The following two time constants

$$\tau_2' \triangleq \tau_1(0) = \frac{2m^2}{\sigma} \text{ and } \tau_1^* = \tau_1(\ell^*) \tag{39}$$

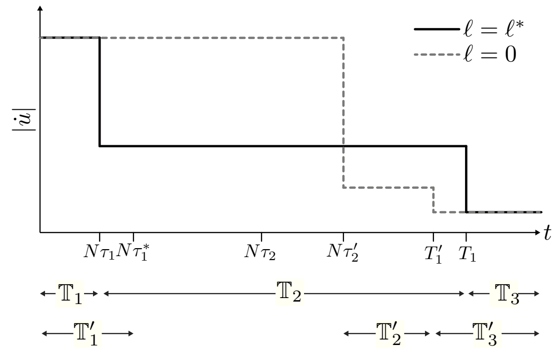

Fig. 1. Transient bounds for $\dot{u}$.

are used to describe the three time intervals that will be used in the analysis of $\dot{u}$ for the ORM case

$$\begin{aligned}\mathbb{T}_1' &= [0, N\tau_1^*)\\ \mathbb{T}_2' &= [N\tau_2', T_1')\\ \mathbb{T}_3' &= [T_1', \infty).\end{aligned} \tag{40}$$

where $T_1' \triangleq \max\{N\tau_2', T(\epsilon, 0)\}$ where $T(\epsilon, 0)$ is from Corollary 2.

As in Definition 3, here too, T exists but is unknown. While these periods for both CRM and ORM are indicated in Figure 1, one cannot apriori conclude if T_1 is greater than or smaller than T_1'. The time instants indicated as in Figure 1 are meant to be merely sketches.

Proposition 13. *Let*

$$\rho_0 \triangleq \frac{\gamma}{\sigma}. \tag{41}$$

For the adaptive system with the classical MRAC given by Eqs (1), (2), (5), (17)–(18) *and* (13)–(14) *with $\ell = 0$, it can be shown that*

$$\begin{aligned}\sup_{t\in\mathbb{T}_1'}|\dot{u}(t)| &\leq \rho_0 d_1 + \sqrt{\rho_0}d_2 + r_1,\\ \sup_{t\in\mathbb{T}_2'}|\dot{u}(t)| &\leq \sqrt{\rho_0}d_3 + d_4 + \sqrt{\frac{1}{\rho_0}}d_5 + \epsilon_1\mathfrak{M}_1(\rho_0, \sqrt{\rho_0}) + r_1\\ \sup_{t\in\mathbb{T}_3'}|\dot{u}(t)| &\leq \sqrt{\frac{1}{\rho_o}}d_6 + d_7 + \epsilon\mathfrak{M}_2(\rho_o, \sqrt{\rho_0}) + r_1\end{aligned} \tag{42}$$

$d_i > 0$, $i = 1$ to 7 are independent of ρ_0, and $\mathfrak{M}_1(\cdot)$ and $\mathfrak{M}_2(\cdot)$ are globally lipschitz with respect to their arguments

The proof of Proposition 13 follows the same steps as in the proof of Theorem 11 and is therefore omitted.

The bounds in (42) indicate that in the classical ORM, one can only derive a bound for $\dot{u}$ over the period $\mathbb{T}_1'$, $\mathbb{T}_2'$ and $\mathbb{T}_3'$. Unlike the CRM case, the procedure in Appendix G cannot be used to derive satisfactory bounds for $\dot{u}$ over $[N\tau_1^*, N\tau_2')$. It also can be seen that unlike the CRM case, τ_2' is fixed and cannot be changed with ℓ. These points are summarized below.

(B1) Over $\mathbb{T}_1'$, $|\dot{u}(t)|$ is bounded by a linear function of ρ_0 and $\sqrt{\rho_0}$

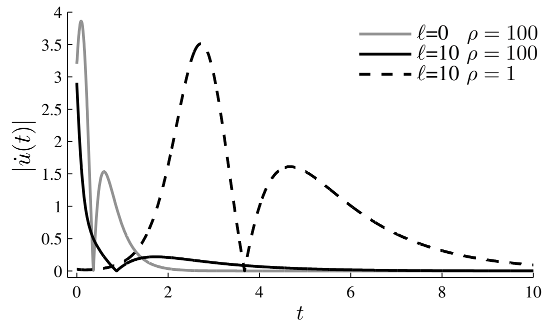

Fig. 2. Plot of $|\dot{u}(t)|$.

(B2) Over $\mathbb{T}_2'$, $|\dot{u}(t)|$ is bounded by a linear function of $\sqrt{\rho_0}$ and $\sqrt{\frac{1}{\rho_0}}$

(B3) Over $\mathbb{T}_3'$, $|\dot{u}(t)|$ is bounded by a linear function of $\sqrt{\frac{1}{\rho_0}}$

(B4) τ_2' is fixed and unlike τ_1, can not be adjusted.

We now compare the bounds on $\dot{u}$ using observations (A1)–(A3) and (B1)–(B3). In order to have the same basis for comparison, we assume that γ, σ, and ℓ are such that $\rho = \rho_0$ and that both CRM– and ORM–adaptive systems start with the same bound at $t = 0$. As noted above, a tight bound cannot be derived for the ORM-based adaptive system over $[N\tau_1^*, N\tau_2')$. In the best scenario, one can assume that this bound is no larger than that over $[0, N\tau_1^*]$. This allows us to derive the bounds shown in Figure 1. The main observations that one can make from this figure are summarized below:

- Even though at time $t = 0$, both the ORM and CRM have the same bound, since τ_1 can be made much smaller than τ_2', this bound is valid for a much shorter time with the CRM-system than in the ORM–system. This helps us conclude that the initial transients can be made to subside much faster in the former case than the latter, by suitably choosing ℓ.
- The bound on $\dot{u}$ for $\mathbb{T}_2$ with the CRM–adaptive system is however linear in powers of ℓ and hence can be larger than the bound on $\dot{u}$ with the ORM-adaptive system over $\mathbb{T}_2'$.
- The above observations clearly illustrate, if the cost function $U(N\tau_2'; \rho, \ell)$ is minimized then the CRM system will have smoother transients than the ORM. Then, at larger times the error dynamics will asymptotically converges to zero.

F. Water–Bed Effect

The discussions in the preceding sections clearly show that CRM-adaptive systems introduce a trade–off: a fast convergence in $e(t)$ with a reduced $\|\dot{\theta}(t)\|_{L_2}$ occurs at the expense of an increased $e^o(t)$. While an optimal choice of ρ and ℓ can minimize this trade–off, it also implies that a badly chosen ℓ and ρ can significantly worsen the adaptive system performance in terms of $e^o(t)$ and $\dot{u}(t)$. We denote this as the water–bed effect and illustrate it through a simulation. Due to space considerations, the details of the adaptive system are omitted and can be found in [10, Section III.E]. Figure 2 shows the behavior of $\dot{u}(t)$ for the ORM, the optimized CRM, and a poorly chosen CRM. The plots clearly show the water–bed effect for the last case and the improved performance of the optimized CRM over the ORM. The free design parameters are also shown in the figure.

IV. Conclusion

This paper concerns the introduction of a feedback gain L in the reference model. In particular we show that, with CRMs, direct adaptive control structures result in guaranteed transient performance. These are primarily realized using the extra degree of freedom available in the CRM in terms of a feedback gain, and by exploiting exponential convergence properties of the CRM–adaptive system. The main impact of this work is the quantification of transient performance in adaptive systems through the investigation of $\mathcal{L}_2$ norms of the model following error, rate of change of the adaptive parameter and the rate of control input.

V. Acknowledgements

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Appendix A
Projection Operator

The Γ–*Projection Operator* for two vectors $\theta, y \in \mathbb{R}^k$, a convex function $f(\theta) \in \mathbb{R}$ and with symmetric positive definite tuning gain $\Gamma \in \mathbb{R}^{k\times k}$ is defined as

$$\mathrm{Proj}_\Gamma(\theta, y, f) = \begin{cases} \Gamma y - \Gamma \frac{\nabla f(\theta)(\nabla f(\theta))^T}{(\nabla f(\theta))^T \Gamma \nabla f(\theta)} \Gamma y f(\theta) \\ \qquad \text{if } f(\theta) > 0 \wedge y^T \Gamma \nabla f(\theta) > 0 \\ \Gamma y \qquad \text{otherwise} \end{cases} \tag{43}$$

where $\nabla f(\theta) = \left(\frac{\partial f(\theta)}{\partial \theta_1} \cdots \frac{\partial f(\theta)}{\partial \theta_k} \right)^T$. The projection operator was first introduced in [12] with extensions in [13] and for a detailed analysis of Γ–projection see [14].

Appendix B
Properties of matrix exponential

Lemma 14 ([10, Corollary 8])**.** *Any Hurwitz matrix $A_m \in \mathbb{R}^{n\times n}$ with constants a and σ as defined in (12) satisfies the following bound for the matrix exponential*

$$\|\exp(A_m \tau)\| \leq m \exp\left(-\frac{\sigma}{2}\tau\right), \tag{44}$$

where $m = \frac{3}{2}(1+4\kappa)^{n-1}$ and $\kappa = \frac{a}{\sigma}$.

Appendix C
Proof of Theorem 6

Proof: Recall the Lyapunov candidate in (9), Taking its time derivative one has that

$$\dot{V} \leq -\|e\|^2 \leq -\frac{1}{\|P\|}V + \frac{1}{\|P\|\gamma}\tilde{\theta}^2_{\max}.$$

Using the upper bound on P from (15)

$$\dot{V} \leq -\alpha_1 V + \alpha_2 \tag{45}$$

with α_1 defined in (21) and $\alpha_2 \triangleq \frac{\sigma+2\ell}{m^2\gamma}\tilde{\theta}^2_{\max}$. Using the Gronwall Bellman Inequality, (45) implies that

$$V(e,\tilde{\theta}) \leq \left(V(e(0),\tilde{\theta}(0)) - \frac{\alpha_2}{\alpha_1} \right) \exp(-\alpha_1 t) + \frac{\alpha_2}{\alpha_1}. \tag{46}$$

Thus, $e(t)$ exponentially converges to the set defined by the following inequality

$$\lim_{t\to\infty} e(t)^T P e(t) \leq \frac{1}{\gamma}\tilde{\theta}^2_{\max}.$$

Using the bound in Lemma 5(ii) we have that

$$e^T P e \geq \frac{1}{2(s+\ell)}\|e\|^2, \tag{47}$$

then we can conclude that $\lim_{t\to\infty}\|e(t)\|^2 \leq \beta_1 \tilde{\theta}^2_{\max}$ where β_1 is defined in (22). The boundedness of $\theta(t)$ follows from the properties of the Projection Algorithm. □

Appendix D
Proof of Theorem 7

Proof: From (46) and (47), we know that

$$\|e(t)\|^2 \leq k_0 \exp\left(-\frac{\sigma+2\ell}{m^2}t\right) + k_1$$

where

$$\begin{aligned} k_0 &= \frac{2(s+\ell)m^2}{\sigma+2\ell}\|e(0)\|^2 + \frac{2(s+\ell)}{\gamma}\|\tilde{\theta}(0)\|^2 - k_1 \\ k_1 &= \frac{2(s+\ell)}{\gamma}\tilde{\theta}^2_{\max}. \end{aligned} \tag{48}$$

Using the following inequalities

$$\frac{2(s+\ell)m^2}{\sigma+2\ell} \leq \frac{2sm^2}{\sigma} \text{ and } \frac{2(s+\ell)}{\gamma} \leq \frac{2s}{\sigma}\frac{\sigma+\ell}{\gamma}$$

the fact that $\|\tilde{\theta}(0)\| \leq \tilde{\theta}_{\max}$ and the definition of ρ from (23), the result in (25) holds with

$$\kappa_1 = \frac{2sm^2}{\sigma} \text{ and } \kappa_2 = \frac{2s}{\sigma}. \tag{49}$$

Beginning with

$$\begin{aligned} \|e(t)\|^2_{L_2} \leq & \int_0^\infty -\dot{V}(e(t),\tilde{\theta}(t)) \leq V(e(0),\tilde{\theta}(0)) \\ \leq & \frac{m^2}{\sigma+2\ell}\|e(0)\|^2 + \frac{1}{\gamma}\|\tilde{\theta}(0)\|^2, \end{aligned} \tag{50}$$

using the definitions of ρ from (23) and the fact that $\frac{1}{\sigma+2\ell} \leq \frac{1}{\sigma+\ell}$ the bound in (26) holds. □

Appendix E
Proof of Theorem 8

Proof: Using (15), the choice for Γ in (14) and the definition of ρ from (23) we have that $\|\Gamma\|\|P\| \leq m^2\rho$. Using the bounds in (28) and (25) for $\|x_m(t)\|$ and $\|e(t)\|$ the results in (29) follow immediately.

For the $\mathcal{L}_2$ norm we begin by observing that

$$\begin{aligned} \|\dot{\theta}(t)\|^2_{L_2} \leq & \|\Gamma\|^2\|P\|^2\|b\|^2 \sup\|x_m(t)\|^2 \int_0^\infty \|e(t)\|^2 dt \\ & + \|\Gamma\|^2\|P\|^2\|b\|^2 \sup\|e(t)\|^2 \int_0^\infty \|e(t)\|^2 dt. \end{aligned}$$

Taking the supremum of (28) and (25) we have upper bounds for $\sup\|x_m(t)\|^2$ and $\sup\|e(t)\|^2$. The $\mathcal{L}_2$ norm of $e(t)$ is given in (26). □

Appendix F
Proof of Theorem 9

Proof: The dynamics of the CRM and the ORM are given in (2) and (10) respectively and leed to the following

$$\dot{x}_m(t) - \dot{x}^o_m(t) = A_m(x_m(t) - x^o_m(t)) - Le. \tag{51}$$

Given that the reference model will have the same initial condition regardless of being closed or open, we then have that

$$\|x_m(t) - x^o_m(t)\| \leq \int_0^t \exp(-\frac{\sigma}{2}\tau)\ell e(\tau) d\tau \tag{52}$$

where the matrix exponential bound came from (44). Using the Cauchy–Schwartz inequality we have the following bound

$$\|x_m(t) - x_m^o(t)\| \leq \frac{\ell m}{\sqrt{\sigma}} \|e(t)\|_{L_2}. \tag{53}$$

APPENDIX G
PROOF OF THEOREM 11

Taking the time derivative of u in (5)

$$\begin{aligned}\dot{u}(t) = &- b^T P e(t) x^T(t) \gamma I_{n\times n} x(t) \\ &+ \theta^T \left(A_m x(t) + b\left(\tilde{\theta}^T x(t) + r(t)\right)\right) + \dot{r}(t).\end{aligned} \tag{54}$$

Substitution of the upper bound on P from (15), using the definition of a_θ from (33) and the bounds on the reference trajectory from Assumption 3 results in the following bound

$$\begin{aligned}|\dot{u}(t)| \leq &\frac{m^2\gamma}{\sigma + 2\ell} \|b\| \|e(t)\| \|x(t)\|^2 \\ &+ \theta_{\max} \left(a_\theta \|x(t)\| + r_0\right) + r_1.\end{aligned} \tag{55}$$

A. Proof of Theorem 11, $t \in \mathbb{T}_1$

Lemma 15. *[Finite time stability] If r satisfies Assumption 3, then*

$$\|x(t)\| \leq \|e(0)\| \exp\left(a_\theta t\right) + \frac{\|b\| r_0}{a_\theta} \left(\exp(a_\theta t) - 1\right), \tag{56}$$

$t \geq 0$ *where a_θ is defined in* (33).

This Lemma follows from [15, Theorem 8.14].

Using the fact that $x(0) = e(0)$ which follows from Assumption 4, Lemma 15 and the definitions of a_θ and τ_1 we obtain that

$$\sup_{t\in\mathbb{T}_1} \|x(t)\| \leq G_{x,1} \tag{57}$$

where $G_{x,1}$ is defined in (36).

Beginning with (25), taking the square root of the expression and noting that $\sqrt{c_1 + c_2} \leq \sqrt{c_1} + \sqrt{c_2}$ for all $c_1, c_2 > 0$, we obtain

$$\|e(t)\| \leq \sqrt{\kappa_1} \exp\left(-\tfrac{1}{\tau_1} t\right) \|e(0)\| + \sqrt{\frac{\kappa_2}{\rho}} \tilde{\theta}_{\max} \tag{58}$$

where τ_1 is defined in (32). This verifies that

$$\sup_{t\in\mathbb{T}_1} \|e(t)\| \leq G_{e,1} \tag{59}$$

where $G_{e,1}$ is defined in (36) . Using (55), (57), and (59), Theorem 11 for $t \in \mathbb{T}_1$ is proved.

B. Proof of Theorem 11, $t \in \mathbb{T}_2$

From (58) it is easy to see that,

$$\sup_{t > N\tau_1} \|e(t)\| \leq G_{e,2} \tag{60}$$

where $G_{e,2}$ is defined in (36).

From (2) and the bound on $\exp(A_m t)$ in (44), we have that

$$\|x_m(t)\| \leq m \int_0^t \exp\left(-\tfrac{1}{\tau_2}(t-\tau)\right) \left(l\|e(\tau)\| + \|b\|\|r\|\right) d\tau \tag{61}$$

Using the integral transform of LTI systems, the bound for $\exp(A_m)$ from (44), the bound for $\|e(t)\|$ from (58), (61) takes the form

$$\begin{aligned}\|x_m(t)\| \leq &m_1 \|e(0)\| \left(\exp\left(-\tfrac{1}{\tau_2} t\right) - \exp\left(-\tfrac{1}{\tau_1} t\right)\right) \\ &+ \frac{2lm}{\sigma}\sqrt{\frac{2(s+\ell)}{\gamma}} \tilde{\theta}_{\max} \left(1 - \exp\left(-\tfrac{1}{\tau_1} t\right)\right) \\ &+ \frac{2\|b\|m}{\sigma} r_0 \left(1 - \exp\left(-\tfrac{1}{\tau_1} t\right)\right)\end{aligned} \tag{62}$$

where $m_1 \triangleq \frac{2lm^4\sqrt{\frac{2s}{\sigma}}}{\sigma + 2\ell - \sigma m^2}$.

Given that $x = e + x_m$, using (59) and (62) one can conclude that

$$\sup_{t \geq N\tau_1} \|x(t)\| \leq G_{x,2} \tag{63}$$

where $G_{x,2}$ is defined in (36). Using (55), (60), and (63), Theorem 11 for $t \in \mathbb{T}_2$ is proved.

C. Proof of Theorem 11, $t \in \mathbb{T}_3$

$G_{e,3}$ follows from Corollary 2. $G_{x,3}$ follows from (62), where it is noted that $t \geq N\tau_2$, and the fact that $\|x\| \leq \|e\| + \|x_m\|$