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On Adaptive Control With Closed-Loop Reference Models: Transients, Oscillations, and Peaking

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ABSTRACT One of the main features of adaptive systems is an oscillatory convergence that exacerbates with the speed of adaptation. Recently, it has been shown that closed-loop reference models (CRMs) can result in improved transient performance over their open-loop counterparts in model reference adaptive control. In this paper, we quantify both the transient performance in the classical adaptive systems and their improvement with CRMs. In addition to deriving bounds on L-2 norms of the derivatives of the adaptive parameters that are shown to be smaller, an optimal design of CRMs is proposed that minimizes an underlying peaking phenomenon. The analytical tools proposed are shown to be applicable for a range of adaptive control problems including direct control and composite control with observer feedback. The presence of CRMs in adaptive backstepping and adaptive robot control is also discussed. Simulation results are presented throughout this paper to support the theoretical derivations.

INDEX TERMS Adaptive systems, adaptive algorithms, adaptive control, observers, closed-loop reference model.

I. INTRODUCTION

A universal observation in all adaptive control systems is a convergent, yet oscillatory behavior in the underlying errors. These oscillations increase with adaptation gain, and as such, lead to constraints on the speed of adaptation. The main obvious challenge in quantification of transients in adaptive systems stems from their nonlinear nature. A second obstacle is the fact that most adaptive systems possess an inherent trade-off between the speed of convergence of the tracking error and the size of parametric uncertainty. In this paper, we overcome these long standing obstacles by proposing an adaptive control design that judiciously makes use of an underlying linear time-varying system, analytical tools that quantify oscillatory behavior in adaptive systems, and the use of tools for decoupling speed of adaptation from parametric uncertainty.

The basic premise of any adaptive control system is to have the output of a plant follow a prescribed reference model through the online adjustment of control parameters. Historically, the reference models in Model Reference Adaptive Control (MRAC) have been open-loop in nature (see for example, [1], [2]), with the reference trajectory generated by a linear dynamic model, and unaffected by the plant output. The notion of feeding back the model following error into the reference model was first reported in [3] and more

recently in [4]–[11]. Denoting the adaptive systems with an Open-loop Reference Model as ORM-adaptive systems and those with closed-loop reference models as CRM-adaptive systems, the design that we propose in this paper to alleviate transients in adaptive control is CRM-based adaptation.

Following stability of adaptive control systems in the 80s and their robustness in the 90s, several attempts have been made to quantify transient performance (see for example, [12]–[14]). The performance metric of interest in these papers stems from either supremum or L-2 norms of key errors within the adaptive system. In [12] supremum and L-2 norms are derived for the model following error, the filtered model following error and the zero dynamics. In [13] L-2 norms are derived for the the model following error in the context of output feedback adaptive systems in the presence of disturbances and un-modeled dynamics. The authors of [14] analyze the interconnection structure of adaptive systems and discuss scenarios under which key signals can behave poorly.

In addition to references [12]–[14], transient performance in adaptive systems has been addressed in the context of CRM adaptive systems in [4]–[11]. The results in [4], [5] focused on the tracking error, with emphasis mainly on the initial interval where the CRM-adaptive system exhibits fast time-scales. In [6] and [7], transient performance is quantified using a damping ratio and natural frequency type of analysis.

However, assumptions are made that the initial state error is zero and that the closed-loop system state is independent of the feedback gain in the reference model, both of which may not hold in general.

In this paper, we start with CRM adaptive systems as the design candidate, and quantify the underlying transient performance. This is accomplished by deriving L-2 bounds on key signals and their derivatives in the adaptive system. These bounds are then related to the corresponding frequency content using a Fourier analysis, thereby leading to an analytical basis for the observed reduction in oscillations with the use of CRM. It is also shown that in general, a peaking phenomenon can occur with CRM-adaptive systems, which then is shown to be minimized through an appropriate design of the CRM-parameters. Extensive simulation results are provided, illustrating the conspicuous absence of oscillations in CRM-adaptive systems in contrast to their dominant presence in ORM-adaptive systems. The results of this paper build on preliminary versions in [8]–[10] where the bounds obtained were conservative. While all results derived in this paper are applicable to plants whose states are accessible for measurement, we refer the reader to [11] for extensions to output feedback.

This paper also addresses Combined/composite direct and indirect Model Reference Adaptive Control (CMRAC) [15], [16], which is another class of adaptive systems in which a noticeable improvement in transient performance was demonstrated. While the results of these papers established stability of combined schemes, no rigorous guarantees of improved transient performance were provided, and have remained a conjecture [17]. We introduce CRMs into the CMRAC and show how improved transients can be guaranteed. We close this paper with a discussion of CRM and related concepts that appear in other adaptive systems as well, including nonlinear adaptive control [18] and adaptive control in robotics [19].

This paper is organized as follows. Section II contains the basic CRM structure with L-2 norms of the key signals in the system. Section III investigates the peaking in the reference model. Section IV contains the multidimensional states accessible extension. Section V investigates composite control structures with CRM. Section VI explores other forms of adaptive control where closed loop structures appear.

II. CRM-BASED ADAPTIVE CONTROL OF SCALAR PLANTS

Let us begin with a scalar system,

$$\dot{x}_p(t) = a_p x_p(t) + k_p u(t) \quad (1)$$

where $x_p(t) \in \mathbb{R}$ is the plant state, $u(t) \in \mathbb{R}$ is the control input, $a_p \in \mathbb{R}$ is an unknown scalar and only the sign of $k_p \in \mathbb{R}$ is known. We choose a closed-loop reference model as

$$\dot{x}_m(t) = a_m x_m(t) + k_m r(t) - \ell(x(t) - x_m(t)). \quad (2)$$

All of the parameters above are known and scalar, $x_m(t)$ is the reference model state, $r(t)$ is a bounded reference input and $a_m, \ell < 0$ so that the reference model and the subsequent

error dynamics are stable. The open-loop reference model dynamics

$$\dot{x}_m^o(t) = a_m x_m^o(t) + k_m r(t) \quad (3)$$

is the corresponding true reference model that we actually want x_p to converge to.

The control law is chosen as

$$u(t) = \bar{\theta}^T(t) \phi(t) \quad (4)$$

where we have defined $\bar{\theta}^T(t) = [\theta(t) \ k(t)]^T$ and $\phi^T(t) = [x_p(t) \ r(t)]^T$ with an update law

$$\dot{\bar{\theta}} = -\gamma \text{sgn}(k_p) e \phi \quad (5)$$

where $\gamma > 0$ is a free design parameter commonly referred to as the adaptive tuning gain and $e(t) = x_p(t) - x_m(t)$ is the state tracking error. From this point forward we will suppress the explicit time dependence of parameters except for emphasis.

We define the parameter error $\tilde{\theta}(t) = \bar{\theta}(t) - \bar{\theta}^*$, where $\bar{\theta}^* \in \mathbb{R}^2$ satisfies $\bar{\theta}^{*T} = \left[\frac{a_m - a_p}{k_p} \ \frac{k_m}{k_p} \right]^T$. The corresponding closed loop error dynamics are:

$$\dot{e}(t) = (a_m + \ell)e + k_p \tilde{\theta}^T \phi. \quad (6)$$

A. STABILITY PROPERTIES OF CRM-ADAPTIVE SYSTEMS

Theorem 1 establishes the stability of the above adaptive system with the CRM.

Theorem 1: The adaptive system with the plant in (1), with the controller defined by (4), the update law in (5) with the reference model as in (2) is globally stable, $\lim_{t \rightarrow \infty} e(t) = 0$, and

$$\|e\|_{L_2}^2 \leq \frac{1}{|a_m + \ell|} \left(\frac{1}{2} e(0)^2 + \frac{|k_p|}{2\gamma} \tilde{\theta}^T(0) \tilde{\theta}(0) \right). \quad (7)$$

Proof: Consider the Lyapunov candidate function

$$V(e(t), \tilde{\theta}(t)) = \frac{1}{2} e^2 + \frac{|k_p|}{2\gamma} \tilde{\theta}^T \tilde{\theta}.$$

Taking the time derivative of V along the system directions we have $\dot{V} = (a_m + \ell)e^2 \leq 0$. Given that V is positive definite and \dot{V} is negative semi-definite we have that $V(e(t), \tilde{\theta}(t)) \leq V(e(0), \tilde{\theta}(0)) \leq \infty$. Thus V is bounded and this means in turn that e and $\tilde{\theta}$ are bounded, with

$$\|e(t)\|_{L_\infty}^2 \leq 2V(0). \quad (8)$$

Given that r and e are bounded and the fact that $a_m < 0$, the reference model is stable. Thus we can conclude x_m , and therefore x_p , are bounded. Given that $\bar{\theta}^*$ is a constant we can conclude that $\bar{\theta}$ is bounded from the boundedness of $\tilde{\theta}$. This can be compactly stated as $e, x_p, \bar{\theta}, \tilde{\theta} \in \mathcal{L}_\infty$, and therefore all of the signals in the system are bounded.

In order to prove asymptotic stability in the error we begin by noting that $-\int_0^t \dot{V} = V(e(0), \tilde{\theta}(0)) - V(e(t), \tilde{\theta}(t)) \leq V(e(0), \tilde{\theta}(0))$. This in turn can be simplified

as $|a_m + \ell| \int_0^t e(\tau)^2 \leq V(0) \forall t \geq 0$. Dividing by $|a_m + \ell|$ and taking the limit as $t \rightarrow \infty$ we have

$$\|e\|_{L_2}^2 \leq \frac{V(0)}{|a_m + \ell|} \quad (9)$$

which implies (7). Given that $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e} \in \mathcal{L}_\infty$, Barbalat's Lemma is satisfied and therefore $\lim_{t \rightarrow \infty} e(t) = 0$ [1]. ■

Theorem 1 clearly shows that CRM ensures stability of the adaptive system. Also, from the fact that $e \in \mathcal{L}_2$ we have that $x_m(t) \rightarrow x_m^o(t)$ as $t \rightarrow \infty$ [20, §IV.1, Theorem 9(c)]. That is, the closed-loop reference model asymptotically converges to the open-loop reference model, which is our true desired trajectory. The questions that arise then is one, if any improvement is possible in the transient response with the inclusion of ℓ , and second, how close is $x_m(t)$, the response of the CRM in relation to that of the original reference model, $x_m^o(t)$. These are addressed in the following section.

B. TRANSIENT PERFORMANCE OF CRM-ADAPTIVE SYSTEMS

The main impact of the CRM is a modification in the realization of the reference trajectory, from the use of a linear model to a nonlinear model. This in turn produces a more benign target for the adaptive closed-loop system to follow, resulting in better transients. It could be argued that the reference model meets the closed-loop system half-way, and therefore reduces the burden of tracking on the adaptive system and shifts it partially to the reference model. In what follows, we precisely quantify this effect.

As Equation (7) in Theorem 1 illustrates, the L-2 norm of e has two components, one associated with the initial error in the reference model, $e(0)$, and the other with the initial error in the parameter space, $\tilde{\theta}(0)$. The component associated with $\tilde{\theta}(0)$ is inversely proportional to the product $\gamma |\ell|$ and the component associated with the initial model following error $e(0)$ is inversely proportional to $|\ell|$ alone. Therefore, without the use of the feedback gain ℓ it is not possible to uniformly decrease the L-2 norm of the model following error. This clearly illustrates the advantage of using the CRM over the ORM, as in the latter, $\ell = 0$.

While CRM-adaptive systems bring in this obvious advantage, they can also introduce an undesirable peaking phenomenon. In what follows, we introduce a definition and show how through a proper choice of the gain ℓ , this phenomenon can be contained, and lead to better bounds on the parameter derivatives. As mentioned in the introduction, we quantify transient performance in this paper by deriving L-2 bounds on the parameter derivative $\dot{\theta}$, which in turn will correlate to bounds on the amplitude of frequency oscillations in the adaptive parameters. For this purpose, we first discuss the L-2 bound on e and supremum bound for x_m . We then describe a peaking phenomenon that is possible with CRM-adaptive systems.

1) L- ∞ NORM OF x_m

The solution to the ODE in (2) is

$$x_m(t) = e^{a_m t} x_m(0) + \int_0^t e^{a_m(t-\tau)} r(\tau) d\tau - \ell \int_0^t e^{a_m(t-\tau)} e(\tau) d\tau. \quad (10)$$

The solution to the ODE in (3) is

$$x_m^o(t) = e^{a_m t} x_m^o(0) + \int_0^t e^{a_m(t-\tau)} r(\tau) d\tau. \quad (11)$$

For ease of exposition and comparison, $x_m(0) = x_m^o(0)$ and thus

$$x_m(t) = x_m^o(t) - \ell \int_0^t e^{a_m(t-\tau)} e(\tau) d\tau. \quad (12)$$

Denoting the difference between the open-loop and closed-loop reference model as $\Delta x_m = x_m - x_m^o$, using Cauchy Schwartz Inequality on $\int_0^t e^{a_m(t-\tau)} \|e(\tau)\| d\tau$, and the bound for $\|e(t)\|_{L_2}$ from (9), we can conclude that

$$\|\Delta x_m(t)\| \leq |\ell| \sqrt{\frac{1}{|2a_m|}} \sqrt{\frac{V(0)}{|a_m + \ell|}}. \quad (13)$$

We quantify the peaking phenomenon through the following definition.

Definition 1: Let $\alpha \in \mathbb{R}^+$, a_1, a_2 are fixed positive constants, $x : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $x(\alpha; t) \in \mathcal{L}_2$,

$$y(\alpha; t) \triangleq \alpha \int_0^t e^{-a_1(t-\tau)} x(\alpha; \tau) d\tau,$$

Then the signal $y(\alpha; t)$ is said to have a peaking exponent s with respect to α if

$$\|y(t)\|_{L_\infty} \leq a_1 \alpha^s + a_2.$$

Remark 1: We note that this definition of peaking differs from that of peaking for linear systems given in [21], and references there in. In these works, the underlying peaking behavior corresponds to terms of the form $\kappa e^{-\alpha t}$, $\alpha, \kappa > 0$, where any increase in α is accompanied by a corresponding increase in κ leading to peaking. This can occur in linear systems where the Jacobian is defective [22]. In contrast, the peaking of interest in this paper occurs in adaptive systems where efforts to decrease the L_2 norm of x through the increase of α leads to the increase of y causing it to peak. This is discussed in detail below.

From Eq. (12), it follows that Δx_m can be equated with y and e with x in Definition 1. Expanding $V(0)$, the bound on $\Delta x_m(t)$ in (13) can be represented as

$$\|\Delta x_m(t)\| \leq b_1 |\ell|^{1/2} + b_2 \left(\frac{|\ell|}{\gamma} \right)^{1/2}$$

where $b_1 = \sqrt{\frac{e(0)^2}{2|a_m|}}$ and $b_2 = \sqrt{\frac{\|\tilde{\theta}(0)\|^2}{2|a_m|}}$. We note that γ is a free design parameter in the adaptive system. Therefore, one can choose $\gamma = |\ell|$ and achieve the bound

$$\|\Delta x_m(t)\| \leq b_1 |\ell|^{1/2} + b_2. \quad (14)$$

From (14) and Definition 1, it follows that with $\gamma = O(|\ell|)$, Δx_m has a peaking exponent of 0.5 with respect to $|\ell|$. Similar to (14) the following bound holds for x_m :

$$\|x_m(t)\|_{L_\infty} \leq b_1|\ell|^{1/2} + b_3 \quad (15)$$

where $b_3 = b_2 + \|x_m^o(t)\|_{L_\infty}$ and $\gamma = |\ell|$. That is the bounds in (14) and (15) increase with $|\ell|$, which implies that $\Delta x_m(t)$ and therefore $x_m(t)$ can exhibit peaking.

While it is tempting to simply pick $e(0) = 0$ so that $b_1 = 0$, as is suggested in [6], [7] to circumvent this problem, it is not always possible to do so, as $x(0)$ may not be available as a measurement because of noise or disturbance that may be present. In Section III, we present an approach where tighter bounds for $x_m(t)$ are derived, which enables us to reduce the peaking exponent of Δx_m from 0.5 to zero.

Before moving to the L-2 bounds on \dot{k} and $\dot{\theta}$, we motivate the importance of L-2 bounds on signal derivatives and how they relate to the frequency characteristic of the signals of interest. We use a standard property of Fourier series and continuous functions [23], [24] summarized in Lemma 1 and Theorem 2 below.

Lemma 1: Consider a periodic signal $f(t) \in \mathbb{R}$ over a finite interval $\mathbb{T} = [t_1, t_1 + \tau]$ where τ is the period of $f(t)$. The Fourier coefficients of $f(t)$ are then given by $F(n) = \frac{1}{\tau} \int_{\mathbb{T}} f(t) e^{-i\omega(n)t} dt$ with $\omega(n) = 2\pi n/\tau$. If $f(t) \in C^1$, given $\epsilon_1 > 0$, there exists an integer N_1 such that

$$(i) \quad \left\| f(t) - \sum_{n=-N_1}^{N_1} F(n) e^{i\omega(n)t} \right\| \leq \epsilon_1, \quad \forall t \in \mathbb{T}.$$

If in addition $f(t) \in C^2$, then for all $\epsilon_2 > 0$ there exists an integer N_2 such that

$$(ii) \quad \left\| \frac{d}{dt} f(t) - \sum_{n=-N_2}^{N_2} i\omega(n) F(n) e^{i\omega(n)t} \right\| \leq \epsilon_2, \quad \forall t \in \mathbb{T}.$$

Remark 2: We note that one can use the notion of generalized functions as presented in [25] in order to obtain Fourier approximations with assumptions of the interval being finite and periodicity relaxed.

Theorem 2: If $f(t) \in C^2$ and periodic with period τ , then the following equality holds

$$\int_{\mathbb{T}} \|\dot{f}(t)\|^2 dt = \sum_{n=-\infty}^{\infty} |F(n)|^2 |\omega(n)2\pi n| \quad (16)$$

where $F(n) = \frac{1}{\tau} \int_{\mathbb{T}} f(t) e^{-i\omega(n)t} dt$, $\omega(n) = 2\pi n/\tau$ and $\mathbb{T} = [t_1, t_1 + \tau]$.

Proof: This follows from Parseval's Theorem. From Lemma 1(ii), $\int_{\mathbb{T}} \|\dot{f}(t)\|^2 dt = \int_{\mathbb{T}} \left\| \sum_{n=-\infty}^{\infty} i\omega(n) F(n) e^{i\omega(n)t} \right\|^2 dt$. Using the orthogonality of $e^{i\omega(n)t}$ we have that $\int_{t_1}^{t_1+\tau} e^{i\omega(n)t} e^{-i\omega(m)t} dt = 0$ for all integers $m \neq n$. It also trivially holds that $\int_{t_1}^{t_1+\tau} e^{i\omega(n)t} e^{-i\omega(n)t} dt = \tau$. Using these two facts along with the fact that the convergence in Lemma 1(ii) is uniform, the integral above can be simplified as $\int_{\mathbb{T}} \|\dot{f}(t)\|^2 dt = \sum_{n=-\infty}^{\infty} \omega(n)^2 |F(n)|^2 \tau$. Expanding one of the $\omega(n)$ terms and canceling the τ term gives us (16).

Remark 3: From Theorem 2 it follows that when the L-2 norm of the derivative of a function is reduced, the product

$|F(n)|^2 |\omega(n)2\pi n|$ is reduced for all $n \in \mathbb{Z}$. Given that $\omega(n)$ is the natural frequency for each Fourier approximation and $|F(n)|$ their respective amplitudes, reducing the L-2 norm of the derivative of a function implicitly reduces the amplitude of the high frequency oscillations. ■

2) L-2 NORM OF $\dot{k}, \dot{\theta}$

With the bounds on e and x_m in the previous sections, we now derive bounds on the adaptive parameter derivatives. From (5) we can deduce that $\|\dot{k}\|^2 = \gamma^2 e^2 r^2$. Integrating both sides and taking the supremum of r we have

$$\int_0^t \|\dot{k}(\tau)\|^2 d\tau \leq \gamma^2 \|r(t)\|_{L_\infty}^2 \|e(t)\|_{L_2}^2. \quad (17)$$

Using the bound on $\|e\|_{L_2}$ from (9) we have that

$$\|\dot{k}(t)\|_{L_2}^2 \leq \frac{2\gamma^2 \|r(t)\|_{L_\infty}^2 V(0)}{|a_m + \ell|}. \quad (18)$$

Similarly, from (5) we can derive the inequality

$$\int_0^t \|\dot{\theta}(\tau)\|^2 d\tau \leq 2\gamma^2 \|e(t)\|_{L_\infty}^2 \int_0^t e(\tau)^2 d\tau + 2\gamma^2 \|x_m(t)\|_{L_\infty}^2 \int_0^t e(\tau)^2 d\tau. \quad (19)$$

Using the bounds for $\|e(t)\|_{L_\infty}$ in (8), $\|e(t)\|_{L_2}$ in (9), and the following bound on

$$\|x_m(t)\|_{L_\infty}^2 \leq 2\|x_m^o(t)\|_{L_\infty}^2 + \frac{|\ell|^2 V(0)}{|a_m| |a_m + \ell|}, \quad (20)$$

which follows from the bound on $\Delta x_m(t)$ in (13), the bound in (19) can be simplified as

$$\|\dot{\theta}(t)\|_{L_2}^2 \leq 4\gamma^2 \frac{V(0)\|x_m^o(t)\|_{L_\infty}^2}{|a_m + \ell|} + 4\gamma^2 \frac{V(0)^2}{|a_m + \ell|} + 2\gamma^2 \frac{|\ell|^2 V(0)^2}{|a_m| |a_m + \ell|^2}. \quad (21)$$

From (18) it is clear that by increasing $|\ell|$ one can arbitrarily decrease the L-2 norm of \dot{k} . The same is not true, however, for the L-2 norm of $\dot{\theta}$ given in (21). Focusing on the first two terms we see that their magnitude is proportional to $\gamma^2/|a_m + \ell|$. Letting ℓ approach negative infinity, the first and second terms in (21) approaches zero and the third term converges to a bound which is proportional to $\gamma^2 V(0)^2$. When $\ell = 0$, the second term becomes proportional to $\gamma^2 V(0)^2$ and the last term in (21) becomes zero. From the previous discussion it is clear that regardless of our choice of ℓ , the only way to uniformly decrease the L-2 norms of the derivatives of the adaptive terms is by decreasing γ . This leads to the classic trade-off present in adaptive control. One can reduce the high frequency oscillations in the adaptive parameters by choosing a small γ , this however leads to poor reference model tracking. This can be seen by expanding the bound on $\|e(t)\|_{L_\infty}$ in (8),

$$\|e(t)\|_{L_\infty}^2 \leq e(0)^2 + \frac{1}{\gamma} \tilde{\theta}(0)^T \tilde{\theta}(0). \quad (22)$$

If one chooses a small γ , then poor state tracking performance can occur, as the second term in (22) is large for small γ . Therefore it still remains to be seen as to how and when CRM leads to an advantage over ORM. As shown in the following subsection and subsequent section, this can be demonstrated through the introduction of projection in the adaptive law and a suitable choice of ℓ and γ . This in turn will allow the reduction of high frequency oscillations.

C. EFFECT OF PROJECTION ALGORITHM

It is well known that some sort of modification of the adaptive law is needed to ensure boundedness in the presence of perturbations such as disturbances or unmodeled dynamics. We use a projection algorithm [26] with CRMs as

$$\dot{\bar{\theta}} = \text{Proj}_{\Omega}(-\gamma \text{sgn}(k_p) e \phi, \bar{\theta}) \quad (23)$$

where $\bar{\theta}(0), \bar{\theta}^* \in \Omega$, with $\Omega \in \mathbb{R}^2$ a closed and convex set centered at the origin whose size is dependent on a known bound of the parameter uncertainty $\bar{\theta}^*$. Equation (23) assures that $\bar{\theta}(t) \in \Omega \forall t \geq 0$ [26]. The following definition will be used throughout:

$$\Theta_{\max} \triangleq \sup_{\bar{\theta}, \bar{\theta}^* \in \Omega} \|\tilde{\theta}\|. \quad (24)$$

Beginning with the already proven fact that $\dot{V} \leq (a_m + \ell)e^2$, we note that the following bound holds as well with the use of (23):

$$\dot{V}(t) \leq -2|a_m + \ell|V + \frac{|a_m + \ell|}{\gamma} |k_p| \Theta_{\max}^2. \quad (25)$$

Using Gronwall-Bellman [27] it can be deduced that

$$V(t) \leq \left(V(0) - \frac{|k_p|}{2\gamma} \Theta_{\max}^2 \right) e^{-2|a_m + \ell|t} + \frac{|k_p|}{2\gamma} \Theta_{\max}^2 \quad (26)$$

which can be further simplified as

$$V(t) \leq \frac{1}{2} \|e(0)\|^2 e^{-2|a_m + \ell|t} + \frac{|k_p|}{2\gamma} \Theta_{\max}^2 \quad (27)$$

which informs the following exponential bound on $e(t)$:

$$\|e(t)\|^2 \leq \|e(0)\|^2 e^{-2|a_m + \ell|t} + \frac{|k_p|}{\gamma} \Theta_{\max}^2. \quad (28)$$

The discussions in Section II show that with a projection algorithm, the CRM adaptive system is not only stable but satisfies the transient bounds in (9), (15), (18), (21), (27) and (28). The bounds in (15), (18) and (21) leave much to be desired however, as it is not clear how the free design parameters ℓ and γ can be chosen so that the bounds on $\|\dot{k}\|_{L_2}$ and $\|\dot{\theta}\|_{L_2}$ can be systematically reduced while simultaneously controlling the peaking in the reference model output x_m . Using the bounds in (27) and (28), in the following section, we propose an ‘‘optimal’’ CRM design that does not suffer from the peaking phenomena, and show how the bounds in (15), (18) and (21) can be further improved. We also make a direct connection between the L-2 norm of the derivative of a signal, and the frequency and amplitude of oscillation in that signal.

III. BOUNDED PEAKING WITH CRM ADAPTIVE SYSTEMS

A. BOUNDS ON x_m

We first show that the peaking that $\|x_m(t)\|$ was shown to exhibit in Section II-A can be reduced through the use of a projection algorithm in the update law as in (23), and a suitable choice of γ and ℓ . For this purpose we derive two different bounds, one over the time interval $[0, t_1]$ and another over $[t_1, \infty)$.

Lemma 2: Consider the adaptive system with the plant in (1), with the controller defined by (4), the update law in (23) with the reference model as in (2). For all $\delta > 1$ and $\epsilon > 0$, there exists a time $t_1 \geq 0$ such that

$$\begin{aligned} \|x_p(t)\| &\leq \delta \|x_p(0)\| + \epsilon \|r(t)\|_{L_\infty} \\ \|x_m(t)\| &\leq \delta \|x_p(0)\| + \epsilon \|r(t)\|_{L_\infty} + \sqrt{2V(0)} \end{aligned} \quad (29)$$

$\forall 0 \leq t \leq t_1$.

Proof: The plant in (1) is described by the dynamical equation

$$\dot{x}_p = (a_m + k_p \tilde{\theta})x_p + k_p \tilde{k}r$$

where we note that $(a_m + k_p \tilde{\theta})$ can be positive. This leads to the inequality

$$\begin{aligned} \|x_p(t)\| &\leq \|x_p(0)\| e^{(a_m + |k_p| \Theta_{\max})t} \\ &\quad + \int_0^t e^{(a_m + |k_p| \Theta_{\max})(t-\tau)} |k_p| \Theta_{\max} \|r(\tau)\|_{L_\infty} d\tau. \end{aligned}$$

For any $\delta > 1$ and any $\epsilon > 0$, it follows from the above inequality that a t_1 exists such that $e^{(a_m + |k_p| \Theta_{\max})t} \leq \delta$ and $\int_0^t e^{(a_m + |k_p| \Theta_{\max})(t-\tau)} |k_p| \Theta_{\max} d\tau \leq \epsilon$, $\forall t \leq t_1$. The bound on $x_m(t)$ follows from the fact that $\|x_m\| \leq \|x_p\| + \|e\|$ and from (8). ■

Remark 4: The above lemma illustrates the fact that if t_1 is small, the plant and reference model states cannot move arbitrarily far from their respective initial conditions over $[0, t_1]$.

Lemma 3: For any $a \geq 0 \exists$ an $x^* < 0$ such that for all $x \leq x^* < 0$

$$e^{xa} \leq |x|^{-\frac{1}{2}} \quad \forall x \leq x^* < 0.$$

Proof: Exponential functions with negative exponent decay faster than any fractional polynomial. ■

We now derive bounds on $x_m(t)$ when $t \geq t_1$. For this purpose a tighter bound on the error e than that in (9) is first derived.

Lemma 4: Consider the adaptive system with the plant in (1), with the controller defined by (4), the update law in (23) with the reference model as in (2). Given a time $t_1 \geq 0$, there exists an ℓ^* s.t.

$$\sqrt{\int_{t_1}^{\infty} \|e\|^2 d\tau} \leq \frac{\|e(0)\|}{\sqrt{2|a_m + \ell|}} + \sqrt{\frac{|k_p|}{2\gamma|a_m + \ell|}} \Theta_{\max} \quad (30)$$

for all $\ell \leq \ell^*$.

Proof: Substitution of $t = t_1$ in (27) and using the fact that $\dot{V}(t) = -|a_m + \ell|\|e(t)\|^2$, the following bound is

obtained:

$$\int_{t_1}^{\infty} \|e\|^2 d\tau \leq \frac{\|e(0)\|^2 e^{-2|a_m + \ell|t_1}}{2|a_m + \ell|} + \frac{|k_p|}{2\gamma|a_m + \ell|} \Theta_{\max}^2. \quad (31)$$

Noting that $\sqrt{e^{-2|a_m + \ell|t_1}} = e^{-|a_m + \ell|t_1}$, and using the result from Lemma 3, we know that there exists an ℓ^* such that for all $\ell < \ell^*$, $e^{(a_m + \ell)t_1} \leq |a_m + \ell|^{-1/2}$. This leads to (30). ■

Similar to the definition of $\Delta x_m(t)$ in Section II.B we define

$$\Delta \bar{x}_m(t) \triangleq |\ell| \int_{t_1}^t e^{-|a_m|(t-\tau)} \|e(\tau)\| d\tau$$

for all $t \geq t_1$. Choosing $\ell \leq \ell^*$ with ℓ^* defined in Lemma 4, using the bound on $e(t)$ in (30) and the Cauchy Schwartz inequality, we have that

$$\|\Delta \bar{x}_m(t)\|_{L_\infty} \leq b_4 + b_5 \left(\frac{|\ell|}{\gamma}\right)^{1/2} \quad \forall t \geq t_1$$

where $b_4 = \frac{\|e(0)\|}{2\sqrt{|a_m|}}$ and $b_5 = \frac{\sqrt{|k_p|} \Theta_{\max}}{2\sqrt{|a_m|}}$. Choosing $\gamma = |\ell|$ the bound above becomes

$$\|\Delta \bar{x}_m(t)\|_{L_\infty} \leq b_4 + b_5 \quad t \geq t_1. \quad (32)$$

Comparing the bound in (32) to the bound in (14), we note that the peaking exponent (Definition 1) has been reduced from 1/2 to 0 for the upper bound on the convolution integral of interest. Thus, as $|\ell|$ is increased, the term $\Delta \bar{x}_m(t)$ will not exhibit peaking. That is, the response of the CRM is fairly close to that of the ORM. This result allows us to obtain a bound on the closed-loop reference model $x_m(t)$ that does not increase with increasing $|\ell|$. This is explored in the following theorem and subsequent remark in detail.

Theorem 3: Consider the adaptive system with the plant in (1), the controller defined by (4), the update law in (23) with the reference model as in (2), with t_1 chosen as in Lemma 2 and $\ell \leq \ell^*$ where ℓ^* satisfies (30). It can then be shown that

$$\|x_m(t)\|_{t \geq t_1}^2 \leq c_1(t) + \frac{\|e(0)\|^2}{|a_m|} + \frac{|\ell| |k_p| \Theta_{\max}^2}{\gamma |a_m|} \quad (33)$$

where

$$c_1 \triangleq 2 \left(\|x_m^o\|_{L_\infty} + \|x_m(t_1)\| e^{-|a_m|(t-t_1)} \right)^2.$$

Proof: The solution of (2) for $t \geq t_1$ is given by

$$\|x_m(t)\| \leq \|x_m^o\|_{L_\infty} + \|x_m(t_1)\| e^{-|a_m|(t-t_1)} + |\ell| \int_{t_1}^{\infty} e^{-|a_m|(t-\tau)} \|e(\tau)\| d\tau.$$

Using the Cauchy Schwartz Inequality and (30) from Lemma 4, we have that

$$\|x_m(t)\|_{t \geq t_1} \leq \|x_m^o\|_{L_\infty} + \|x_m(t_1)\| e^{-|a_m|(t-t_1)} + \frac{|\ell|}{\sqrt{2|a_m|}} \cdot \left(\frac{\|e(0)\|}{\sqrt{2|a_m + \ell|}} + \frac{\sqrt{|k_p|} \Theta_{\max}}{\sqrt{2\gamma|a_m + \ell|}} \right),$$

for all $\ell < \ell^*$. Squaring leads to (33). ■

Corollary 4: Following the same assumptions as Theorem 3, with $\gamma = |\ell|$

$$\|x_m(t)\|_{t \leq t_1}^2 \leq 2(\delta \|x_p(0)\| + \epsilon \|r(t)\|_{L_\infty})^2 + 4V(0) \quad (34)$$

$$\|x_m(t)\|_{t \geq t_1}^2 \leq c_1(t) + \frac{\|e(0)\|^2}{|a_m|} + \frac{|k_p| \Theta_{\max}^2}{|a_m|} \quad (35)$$

Remark 5: Through the use of a projection algorithm in the adaptive law, the exploitation of finite time stability of the plant in Lemma 1, and through the use of the extra degree of freedom in the choice of ℓ , we have obtained a bound for $\|x_m(t)\|_{t \leq t_1}^2$ in (34) which is only a function of the initial condition of the plant and controller. Similarly for $\|x_m(t)\|_{t \geq t_1}^2$, we have derived a bound in (34) which is once again a function of the initial condition of the plant and controller alone. The most important point to note is that unlike (15), the bound on x_m in (34) and (35) is no longer proportional to ℓ in any power. This implies that even for large $|\ell|$, an appropriate choice of the adaptive tuning parameter γ can help reduce the peaking in the reference model. This improvement was possible only through the introduction of projection and the use of the Gronwall-Bellman inequality.

B. BOUNDS ON PARAMETER DERIVATIVES AND OSCILLATIONS

We now present the main result of this paper.

Theorem 5: The adaptive system with the plant in (1), the controller defined by (4), the update law in (23) with the reference model as in (2), with t_1 chosen as in Lemma 2 and $\ell \leq \ell^*$ where ℓ^* is given in Lemma 4 and $\gamma = |\ell|$, the following bounds are satisfied for all $\gamma \geq 1$:

$$\begin{aligned} \int_{t_1}^{\infty} \|\dot{k}\|^2 d\tau &\leq \left(\|e(0)\|^2 + |k_p| \Theta_{\max}^2 \right) \|r(t)\|_{L_\infty} \\ \int_{t_1}^{\infty} \|\dot{\theta}\|^2 d\tau &\leq \left(\|e(0)\|^2 + |k_p| \Theta_{\max}^2 \right) c_2 \\ &\quad + \left(\|e(0)\|^2 + |k_p| \Theta_{\max}^2 \right) \left(\frac{c_3}{\sqrt{|\ell|}} + \frac{c_4}{\gamma} \right) \end{aligned} \quad (36)$$

where c_2, c_3, c_4 are independent of γ and ℓ , and are only a function of the initial conditions of the system and the fixed design parameters.

Proof: Using (17) and (30), together with the fact that $\gamma = |\ell|$, we obtain the first inequality in (36). To prove the bound on $\dot{\theta}$ we start with (19), and note that

$$\gamma^2 \int_{t_1}^{\infty} e(\tau)^2 d\tau \leq \left(\|e(0)\|^2 + |k_p| \Theta_{\max}^2 \right). \quad (37)$$

Using the bound in (37) and setting $c_2 = \|x_m(t)\|_{t \geq t_1}^2$ from (33) we have the first term in the bound on $\dot{\theta}$ in (36).

We note from (28) and Lemma 3 that

$$\|e(t)\| \leq \frac{e(0)^2}{\sqrt{|a_m + \ell|}} + \frac{|k_p| \Theta_{\max}}{\gamma} \quad \forall t \geq t_1.$$

This together with (37) leads to the second term in the bound on $\dot{\theta}$ in (36). Therefore, c_2, c_3 and c_4 are independent of γ and ℓ . ■

Remark 6: From the above Theorem it is clear that if γ and $|\ell|$ are increased while holding $\gamma = |\ell|$, the L-2 norms of the derivatives of the adaptive parameters can be decreased significantly. Two important points should be noted. One is that the bounds in (36) are much tighter than those in (21), with terms of the form γ^2/ℓ no longer present. Finally, from Theorem 2, it follows that the improved L-2 bounds in (36) result in a reduced high frequency oscillations in the adaptive parameters.

Remark 7: Noting the structure of the control input in (4), it follows directly that reduced oscillations in $\theta(t)$ and $k(t)$ results in reduced oscillation in the control input for the following reason. We note that

$$x_p(t) = e^{a_m t} x_p(0) + \int_0^t e^{a_m(t-\tau)} k_p (\tilde{k}r + \tilde{\theta}x_p) d\tau.$$

Since $\tilde{\theta}$ and \tilde{k} have reduced oscillations, $x_p(t)$ will be smooth, resulting in $\theta(t)x_p(t)$ and therefore $u(t)$ to have reduced oscillations. It should also be noted that with $\ell < \ell^*$ and $\gamma = |\ell|$, it follows that $\|x(t)\|_{L_\infty}$ is independent of ℓ .

C. SIMULATION STUDIES FOR CRM

Simulation studies are now presented to illustrate the improved transient behavior of the adaptive parameters and the peaking that can occur in the reference model. For these examples the reference system is chosen such that $a_m = -1$, $k_m = 1$ and the plant is chosen as $a_p = 1$, $k_p = 2$. The adaptive parameters are initialized to be zero. Figures 1 through 3 are for an ORM adaptive system with the tuning gain chosen as $\gamma \in \{1, 10, 100\}$. Walking through Figures 1 through 3 it clear that as the tuning gain is increased the plant tracks the reference model more closely, at the cost of increased oscillations in the adaptive parameters. Then the CRM is introduced and the resulting responses are shown in Figures 4 through 6, for $\gamma = 100$, and $\ell = -10, -100$, and -1000 respectively. First, it should be noted that no high frequency oscillations are present in these cases, and the trajectories are smooth, which corroborates the inequalities (36) in Theorem 5. As the ratio $|\ell|/\gamma$ increases, as illustrated in Figures 4 through 6, the reference trajectory x_m starts to deviate from the open-loop reference x_m^o , with the peaking phenomenon clearly visible in Figure 6 where $|\ell|/\gamma = 10$. This corroborates our results in section III as well.

IV. CRM FOR STATES ACCESSIBLE CONTROL

In this section we show that the same bounds shown previously easily extend to the states accessible case. Consider the n dimensional linear system

$$\dot{x}_p = Ax_p + B\Lambda u \tag{38}$$

with B known, A unknown, and Λ an unknown diagonal matrix of positive elements. An a priori upper bound on Λ is known and therefore we define

$$\bar{\lambda} \triangleq \max_i \lambda_i(\Lambda),$$

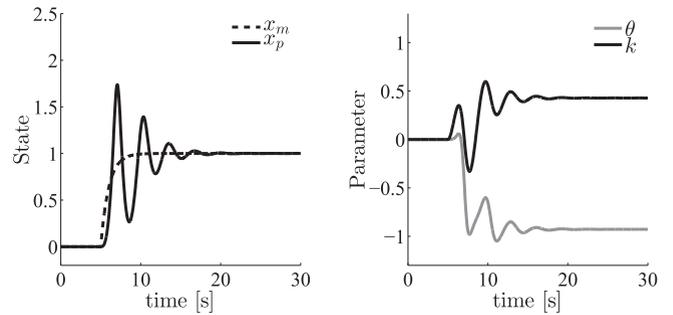


FIGURE 1. Trajectories of the ORM adaptive system $\gamma = 1$.

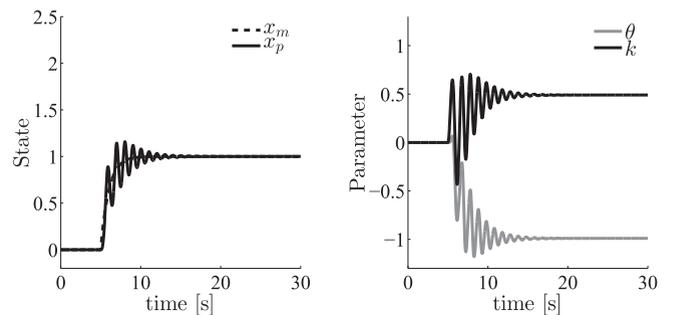


FIGURE 2. Trajectories of the ORM adaptive system $\gamma = 10$.

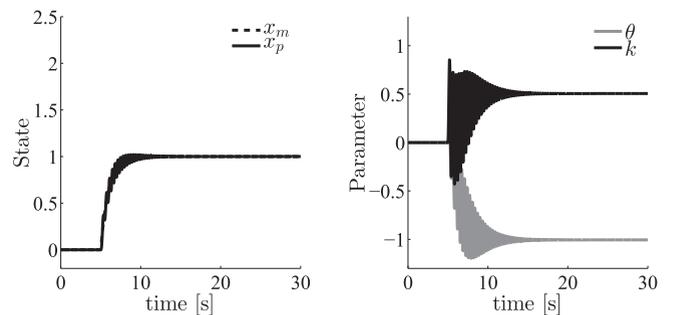


FIGURE 3. Trajectories of the ORM adaptive system $\gamma = 100$.

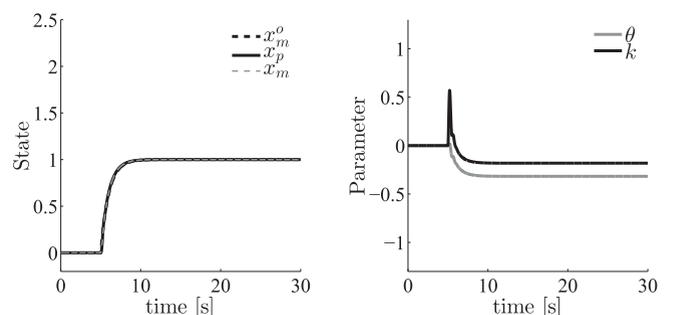


FIGURE 4. Trajectories of the CRM adaptive system $\gamma = 100, \ell = -10$.

where λ_i denotes the i -th Eigenvalue. The reference model is defined as

$$\dot{x}_m = A_m x_m + Br - Le. \tag{39}$$

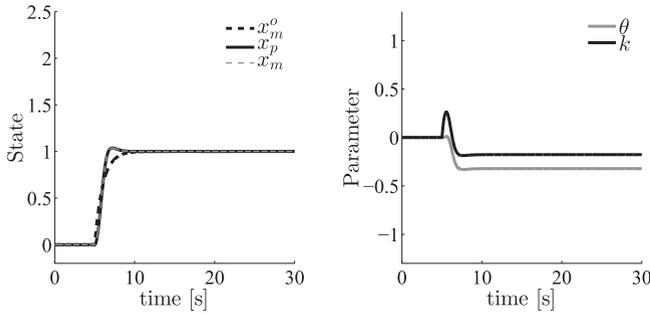


FIGURE 5. Trajectories of the CRM adaptive system $\gamma = 100$, $\ell = -100$.

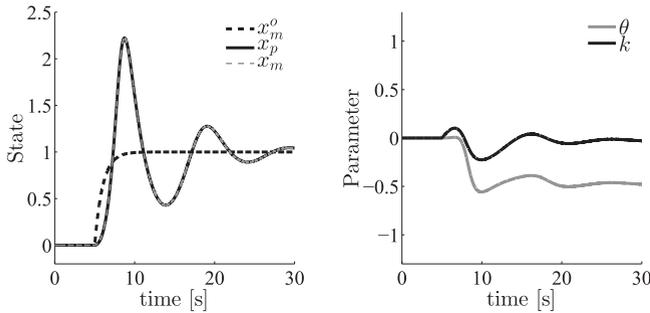


FIGURE 6. Trajectories of the CRM adaptive system $\gamma = 100$, $\ell = -1000$.

The control input is defined as

$$u = \Theta x_p + Kr. \quad (40)$$

It is assumed that there exists Θ^* and K^* such that

$$\begin{aligned} A + B\Lambda\Theta^* &= A_m \\ \Lambda K^* &= I \end{aligned}$$

and the parameter errors are then defined as $\tilde{\Theta} = \Theta - \Theta^*$ and $\tilde{K} = K - K^*$. Defining the error as $e = x_p - x_m$, the update law for the adaptive parameters is then

$$\begin{aligned} \dot{\Theta} &= \text{Proj}_{\Omega_1}(-\Gamma B^T P e x_p^T, \Theta) \\ \dot{K} &= \text{Proj}_{\Omega_2}(-\Gamma B^T P e r^T, K) \end{aligned} \quad (41)$$

where $P = P^T > 0$ is the solution to the Lyapunov equation $(A_m + L)^T P + P(A_m + L) = -Q$ which exists for all $Q = Q^T > 0$. With a slight abuse of notation the following definition is reused from the previous section,

$$\sup_{\Theta, \Theta^* \in \Omega_1} \|\tilde{\Theta}\|_F \triangleq \Theta_{\max} \quad \text{and} \quad \sup_{K, K^* \in \Omega_2} \|\tilde{K}\|_F \triangleq K_{\max}, \quad (42)$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The adaptive system can be shown to be stable by using the following Lyapunov candidate,

$$V(t) = e^T P e + \text{Tr}(\Lambda \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta}) + \text{Tr}(\Lambda \tilde{K}^T \Gamma^{-1} \tilde{K})$$

where after differentiating we have that $\dot{V} \leq -e^T Q e$. We choose L and Γ in a special form to ease the analysis in the following sections.

Assumption 1: The free design parameters are chosen as

$$\begin{aligned} \Gamma &= \gamma I_{n \times n} \\ L &= -A_m + g I_{n \times n} \end{aligned} \quad (43)$$

where $\gamma > 0$ and $g < 0$.

Assumption 1 allows us to choose a $P = 1/2 I_{n \times n}$ in the Lyapunov equation and therefore $Q = -g I_{n \times n}$. Using these simplification the Lyapunov candidate derivative can be bounded as

$$\dot{V}(t) \leq -|g| \|e(t)\|^2, \quad (44)$$

and by direct integration we have

$$\|e(t)\|^2 \leq \frac{V(0)}{|g|}. \quad (45)$$

Using the Gronwall-Bellman Lemma as was previously used in (25)-(28), we can deduce that

$$V(t) \leq \frac{1}{2} \|e(0)\|^2 e^{-2|g|t} + \frac{\bar{\lambda}}{\gamma} \left(\Theta_{\max}^2 + K_{\max}^2 \right). \quad (46)$$

Lemma 5: For all $\epsilon > 0$ and $\delta > 1$ there exists a t_2 such that the plant and reference model in (38) and (39) respectively satisfy the bounds in (29) with t_2 replacing t_1 .

Lemma 6: Consider the adaptive system with the plant in (38), the controller in (40), the update law in (41), the reference model as in (39) and Γ and L parameterized as in Assumption 1. Given a time $t_2 \geq 0$, there exists a g^* s.t.

$$\sqrt{\int_{t_2}^{\infty} \|e\|^2 d\tau} \leq \frac{\|e(0)\|}{\sqrt{2}|g|} + \sqrt{\frac{\bar{\lambda}(\Theta_{\max}^2 + K_{\max}^2)}{\gamma|g|}} \quad (47)$$

for all $g \leq g^*$.

Proof: From (59) we have that

$$V(t_2) \leq \frac{1}{2} \|e(0)\|^2 e^{-2|g|t_2} + \frac{\bar{\lambda}}{\gamma} \left(\Theta_{\max}^2 + K_{\max}^2 \right).$$

Using the above bound and integrating $-\dot{V}$ in (59) from t_2 to ∞ and dividing by $|g|$ leads to

$$\int_{t_2}^{\infty} \|e\|^2 d\tau \leq \frac{\|e(0)\|^2 e^{-2|g|t_2}}{2|g|} + \frac{\bar{\lambda}}{\gamma|g|} \left(\Theta_{\max}^2 + K_{\max}^2 \right) \quad (48)$$

Taking the square root, noting that $\sqrt{e^{-2|g|t_2}} = e^{-|g|t_2}$, and using the result from Lemma 3, we know that there exists an g^* such that for all $g < g^*$, $e^{-|g|t_2} \leq |g|^{-1/2}$. ■

Theorem 6: Consider the adaptive system with the plant in (38), the controller in (40), the update law in (41), the reference model as in (39), Γ and L parameterized as in Assumption 1, with t_2 chosen as in Lemma 5 and $g \leq g^*$ where g^* is given in Lemma 6. It can be shown that

$$\begin{aligned} \|x_m(t)\|_{t \geq t_2}^2 &\leq c_5(t) + 2 \left(\frac{\|A_m\|^2}{|g|^2} + 1 \right) \frac{a_1}{a_2} \|e(0)\|^2 \\ &\quad + 4\bar{\lambda} \left(\frac{\|A_m\|^2}{\gamma|g|^2} + \frac{|g|}{\gamma} \right) \frac{a_1}{a_2} \left(\Theta_{\max}^2 + K_{\max}^2 \right) \end{aligned} \quad (49)$$

where

$$c_5 \triangleq 2 \left(\|x_m^o\|_{L_\infty} + \|x_m(t_2)\| a_1 e^{-a_2(t-t_2)} \right)^2$$

$$e^{A_m t} \leq a_1 e^{a_2 t}$$

with $a_1, a_2 > 0$.

Proof: The existence of $a_1, a_2 > 0$ such that $e^{A_m t} \leq a_1 e^{-a_2 t}$ follows from the fact that A_m is Hurwitz [22]. Consider the dynamical system in (39) for $t \geq t_2$,

$$\|x_m(t)\|_{t \geq t_2} \leq \|x_m^o\|_{L_\infty} + \|x_m(t_1)\| a_1 e^{-a_2(t-t_1)}$$

$$+ \|L\| a_1 \int_{t_2}^{\infty} e^{-a_2(t-\tau)} \|e(\tau)\| d\tau.$$

Using Cauchy Schwartz along with (47) in Lemma 6 we have that

$$\|x_m(t)\|_{t \geq t_2} \leq \|x_m^o\|_{L_\infty} + \|x_m(t_2)\| e^{-|a_m|(t-t_1)}$$

$$+ \frac{\|L\| \sqrt{a_1}}{\sqrt{2a_2}} \left(\frac{\|e(0)\|}{\sqrt{2}|g|} + \sqrt{\frac{\bar{\lambda}(\Theta_{\max}^2 + K_{\max}^2)}{\gamma |g|}} \right).$$

Squaring and using the fact that $L = -A_m + gI_{n \times n}$ and thus $\|L\| \leq \|A_m\| + g$ we have that

$$\|x_m(t)\|_{t \geq t_2}^2 \leq c_5(t) + \frac{(\|A_m\| + g)^2 a_1}{|g|^2 a_2} \|e(0)\|^2$$

$$+ \frac{2(\|A_m\| + g)^2 \bar{\lambda} a_1}{\gamma |g| a_2} \left(\Theta_{\max}^2 + K_{\max}^2 \right).$$

Inequality (49) follows since $(\|A_m\| + |g|)^2 \leq 2\|A_m\|^2 + 2|g|^2$. ■

Remark 8: Just as in the scalar case, we have derived a bound for $\|x_m(t)\|_{t \geq t_2}^2$ which is once again a function of the initial condition of the plant and controller, but also dependent on a component which is proportional to $|g|/\gamma$. Therefore, by choosing $\frac{|g|}{\gamma} = 1$ with $\gamma > 0$ we can have bounded peaking in the reference model.

Theorem 7: The adaptive system with the plant in (38), the controller in (40), the update law in (41), the reference model as in (39), Γ and L parameterized as in Assumption 1, with t_2 chosen as in Lemma 5, $g \leq g^*$ where g^* is given in Lemma 6 and γ chosen such that $\gamma = |g|$ the following bounds are satisfied for all $\gamma \geq 1$:

$$\int_{t_2}^{\infty} \|\dot{K}\|^2 d\tau \leq \|B\| \left(\|e(0)\|^2 + K_{\max}^2 + \Theta_{\max}^2 \right) \|r(t)\|_{L_\infty}$$

$$\int_{t_2}^{\infty} \|\dot{\Theta}\|^2 d\tau \leq \|B\| \left(\|e(0)\|^2 + \Theta_{\max}^2 + K_{\max}^2 \right)$$

$$\cdot \left(c_6 + \frac{c_7}{g^2} + \frac{c_8}{\sqrt{|g|}} + \frac{c_9}{\gamma} \right) \quad (50)$$

where c_6, c_7, c_8, c_9 are independent of γ and g , and are only a function of the initial conditions of the system and the fixed design parameters.

Proof: The proof follows the same steps as used to derive the bounds in Theorem 5. ■

Remark 9: It should be noted that if γ and $|g|$ are increased while holding $\gamma = |g|$, the L-2 norms of the derivatives of the adaptive parameters can be decreased significantly.

Remark 10: The similarity of the bounds in Theorem 7 to those in Theorem 5 implies that the same bounds on frequencies and corresponding amplitudes of the overall adaptive systems as in Theorem 2 hold here in the higher-order plant as well.

We note that robustness issues have not been addressed with the CRM architecture in this work. However, recent results in [28]–[31] have shown that adaptive systems do have a time-delay margin and robustness to unmodeled dynamics when projection is used in the update law. While we expect similar results to hold with CRM as well, a detailed investigation of the same as well as comparisons of their robustness properties to their ORM counterparts are topics for further research.

V. CRM COMPOSITE CONTROL WITH OBSERVER FEEDBACK

In this section, we show that the tools introduced to demonstrate smooth transient in CRM-adaptive systems can be used to analyze CMRAC systems introduced in [15]–[17]. As mentioned in the introduction, these systems were observed to exhibit smooth transient response, and yet no analytical explanations have been provided until now for this behavior. Our focus is on first-order plants for the sake of simplicity. Similar to Section IV, all results derived here can be directly extended to higher order plants whose states are accessible.

The CMRAC system that we discuss in this paper differs from that in [15] and includes an observer whose state is fed back for control rather than the plant state. As mentioned in the introduction, we denote this class of systems as CMRAC-CO and is described by the plant in (1), the reference model in (2), an observer as

$$\dot{x}_o(t) = \ell(x_o - x_p) + (a_m - k_p \hat{\theta})x_o(t) + k_p u(t), \quad (51)$$

and the control input is given by

$$u = \theta x_o + k^* r. \quad (52)$$

In the above k_p is assumed to be known for ease of exposition. The feedback gain ℓ is chosen so that

$$g_\theta \triangleq a_m + \ell + |k_p \theta^*| < 0. \quad (53)$$

Defining $e_m = x_p - x_m$ and $e_o = x_o - x_p$, the error dynamics are now given by

$$\dot{e}_m(t) = (a_m + \ell)e_m + k_p \tilde{\theta} x_o + k_p \theta^* e_o$$

$$\dot{e}_o(t) = (a_m + \ell)e_o - k_p \tilde{\theta} x_o. \quad (54)$$

where $\tilde{\theta} = \theta - \theta^*$ and $\hat{\theta} = \hat{\theta} - \theta^*$ with θ^* satisfying $a_p + k_p \theta^* = a_m$ and $k_p k^* = k_m$. The update laws for the adaptive parameters are then defined with the update law

$$\dot{\theta} = \text{Proj}_\Omega(-\gamma \text{sgn}(k_p) e_m x_o, \theta) - \eta \epsilon_\theta$$

$$\dot{\hat{\theta}} = \text{Proj}_\Omega(\gamma \text{sgn}(k_p) e_o x_o, \hat{\theta}) + \eta \epsilon_\theta$$

$$\epsilon_\theta = \theta - \hat{\theta} \quad (55)$$

where $\gamma, \eta > 0$ are free design parameters. As before we define the bounded set

$$\Theta_{\max} \triangleq \max \left\{ \sup_{\theta, \theta^* \in \Omega} \|\tilde{\theta}\|, \sup_{\hat{\theta}, \theta^* \in \Omega} \|\bar{\theta}\| \right\}. \quad (56)$$

We first establish stability and then discuss the improved transient response.

A. STABILITY

The stability of the CMRAC-CO adaptive system given by (1), (2), (51)-(55) can be verified with the following Lyapunov candidate

$$V(t) = \frac{1}{2} \left(e_m^2 + e_o^2 + \frac{|k_p|}{\gamma} \tilde{\theta}^2 + \frac{|k_p|}{\gamma} \bar{\theta}^2 \right) \quad (57)$$

which has the following derivative

$$\dot{V} \leq g_\theta e_m^2 + g_\theta e_o^2 - \frac{\eta |k_p|}{\gamma} \epsilon_\theta^2. \quad (58)$$

Boundedness of all signals in the system follows since $g_\theta < 0$. From the integration of (58) we have $\{e_m, e_o, \epsilon_\theta\} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and thus $\lim_{t \rightarrow \infty} \{e_m, e_o, \epsilon_\theta\} = \{0, 0, 0\}$. Using the Gronwall-Bellman Lemma as was previously used in (25)-(28), we can deduce that

$$V(t) \leq \frac{1}{2} \left(e_m(0)^2 + e_o(0)^2 \right) e^{-2|g_\theta|t} + \frac{|k_p|}{\gamma} \theta_{\max}^2. \quad (59)$$

It should be noted that the presence of a non-zero ℓ is crucial for stability, as g_θ cannot be guaranteed to be negative if $\ell = 0$.

B. TRANSIENT PERFORMANCE OF CMRAC-CO

Similar to Sections II and III we divide the timeline into $[0, t_3]$ and $[t_3, \infty)$, where t_3 is arbitrarily small. We first derive bounds for the system states over the initial $[0, t_3]$ in Lemma 7, bounds for the tracking, observer, parameter estimation errors e_m, e_o and ϵ_θ over $[t_3, \infty)$ in Lemma 8, bounds for x_o over $[t_3, \infty)$ in Theorem 8, and finally bounds for the parameter derivatives $\dot{\theta}$ and $\hat{\theta}$ in Theorem 10.

Lemma 7: Consider the CMRAC-CO adaptive system with the plant in (1), with the controller defined by (52), the update law in (55) and with the reference model as in (2). For all $\delta > 1$ and $\epsilon > 0$, there exists a time $t_3 \geq 0$ such that

$$\begin{aligned} \|x_p(t)\| &\leq \delta \|x_p(0)\| + \epsilon \left(\|r(t)\|_{L_\infty} + \sqrt{2V(0)} \right) \\ \|x_o(t)\| &\leq \delta \|x_p(0)\| + \epsilon \|r(t)\|_{L_\infty} + (1 + \epsilon) \sqrt{2V(0)} \end{aligned} \quad (60)$$

$\forall 0 \leq t \leq t_3$.

Proof: The plant in (2) with the controller in (52) can be represented as

$$\dot{x}_p = (a_p + k_p \theta) x_p + k_p (r + \theta e_o)$$

where we note that $(a_p + k_p \theta)$ can be positive. This leads to the inequality

$$\begin{aligned} \|x_p(t)\| &\leq \|x_p(0)\| e^{(a_p + |k_p| \Theta_{\max})t} + \int_0^t e^{(a_p + |k_p| \Theta_{\max})(t-\tau)} \\ &\quad \cdot |k_p| \left(\Theta_{\max} \|r(\tau)\| + \Theta_{\max} \|e_o(\tau)\| \right) d\tau. \end{aligned}$$

For any $\delta > 1$ and any $\epsilon > 0$, it follows from the above inequality that a t_3 exists such that $e^{(a_p + |k_p| \Theta_{\max})t} \leq \delta$ and $\int_0^t e^{(a_p + |k_p| \Theta_{\max})(t-\tau)} |k_p| \Theta_{\max} d\tau \leq \epsilon$, $0 \leq t \leq t_3$ given $\delta > 0$ and $\epsilon > 0$. From the structure of the Lyapunov candidate in (57) and the fact that $\dot{V} \leq 0$ we have that $\|e_o(t)\|_{L_\infty} \leq \sqrt{2V(0)}$. The bound on $x_o(t)$ follows from the fact that $\|x_o\| \leq \|x_p\| + \|e_o\|$. ■

Lemma 8: Consider the adaptive system with the plant in (1), the controller in (52), the update law in (55), and the reference model as in (2). Given a time $t_2 \geq 0$, there exists a g_θ^* s.t.

$$\begin{aligned} \sqrt{\int_{t_3}^\infty e_m^2 d\tau} &\leq \frac{\sqrt{e_m(0)^2 + e_o(0)^2}}{\sqrt{2} |g_\theta|} + \sqrt{\frac{|k_p|}{\gamma |g_\theta|}} \Theta_{\max} \\ \sqrt{\int_{t_3}^\infty e_o^2 d\tau} &\leq \frac{\sqrt{e_m(0)^2 + e_o(0)^2}}{\sqrt{2} |g_\theta|} + \sqrt{\frac{|k_p|}{\gamma |g_\theta|}} \Theta_{\max} \\ \sqrt{\int_{t_3}^\infty \epsilon_\theta^2 d\tau} &\leq \frac{\sqrt{\gamma} \sqrt{e_m(0)^2 + e_o(0)^2}}{\sqrt{2\eta} |k_p| |g_\theta|} + \sqrt{\frac{1}{\eta}} \Theta_{\max} \end{aligned} \quad (61)$$

for all $g_\theta \leq g_\theta^*$.

Theorem 8: Consider the adaptive system with the plant in (1), the controller in (52), the update law in (55), the reference model as in (2), with t_3 chosen as in Lemma 7 and $g_\theta \leq g_\theta^*$ where g_θ^* is given in Lemma 8. It can be shown that

$$\begin{aligned} \|x_o(t)\|_{t \geq t_3}^2 &\leq c_{10}(t) + \frac{|\ell|^2}{|g_\theta|^2} 2\sqrt{a_4} \left(e_m(0)^2 + e_o(0)^2 \right) \\ &\quad + \frac{|\ell|^2 |k_p|}{\gamma |g_\theta|} 4a_4 \Theta_{\max}^2 \end{aligned} \quad (62)$$

where

$$\begin{aligned} c_{10} &\triangleq 2 \left(a_3 |k_p| \|r\|_{L_\infty} + \|x_o(t_3)\| e^{a_\theta(t-t_3)} \right)^2 \\ a_\theta &\triangleq a_m + k_p \epsilon_\theta \end{aligned}$$

and

$$\begin{aligned} \int_{t_3}^\infty e^{a_\theta(t-\tau)} d\tau &\leq a_3 \\ \int_{t_3}^\infty e^{2a_\theta(t-\tau)} d\tau &\leq a_4 \end{aligned}$$

with $0 \leq a_i < \infty$, $i \in \{3, 4\}$.

Proof: Given that $\lim_{t \rightarrow \infty} \epsilon_\theta(t) = 0$ we have from (53) that $\lim_{t \rightarrow \infty} a_\theta = a_m$. Thus $\lim_{t \rightarrow \infty} e^{a_\theta t} = 0$. Therefore, $a_3, a_4 < \infty$. Consider the dynamical system in (51) for $t \geq t_3$.

$$\begin{aligned} \|x_o(t)\|_{t \geq t_3} &\leq \|x_o(t_3)\| e^{a_\theta(t-t_3)} \\ &\quad + \int_{t_3}^\infty e^{a_\theta(t-\tau)} \left(|\ell| \|e_o(\tau)\| + |k_p| \|r(\tau)\| \right) d\tau. \end{aligned}$$

Using Cauchy Schwartz and Lemma 8 as before we have

$$\begin{aligned} \|x_o(t)\|_{t \geq t_3} &\leq a_3 |k_p| \|r\|_{L_\infty} + \|x_o(t_3)\| e^{-|a_m|(t-t_3)} \\ &+ |\ell| \sqrt{a_4} \frac{\sqrt{e_m(0)^2 + e_o(0)^2}}{\sqrt{2} |g_\theta|} \\ &+ |\ell| \sqrt{a_4 a_4} \sqrt{\frac{|k_p|}{\gamma |g_\theta|}} \Theta_{\max}. \end{aligned}$$

Squaring and using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ twice, we have our result. ■

Corollary 9: For the system presented in Theorem 8 setting $\gamma = |g_\theta|$ and taking the limit as $\ell \rightarrow -\infty$ the following bound holds for $x_o(t)$

$$\lim_{\ell \rightarrow -\infty} \|x_o(t)\|_{t \geq t_3}^2 \leq c_{10}(t) + 2\sqrt{a_4} (e_m(0)^2 + e_o(0)^2) + |k_p| 4a_4 \Theta_{\max}^2. \quad (63)$$

Theorem 10: The adaptive system with the plant in (1), the controller defined by (52), the update law in (55) with the reference model as in (2), with t_3 chosen as in Lemma 7 and $g_\theta \leq g_\theta^*$ where g_θ^* is given in Lemma 8 and γ chosen such that $\gamma = |g_\theta|$ the following bounds are satisfied for all $\gamma \geq 1$:

$$\int_{t_3}^{\infty} \|\dot{\theta}\|^2 d\tau \leq \alpha \text{ and } \int_{t_3}^{\infty} \|\dot{\hat{\theta}}\|^2 d\tau \leq \alpha \quad (64)$$

with

$$\alpha \triangleq \left(\frac{\ell^2}{g_\theta^2} + \frac{\eta}{|k_p|} \right) (e_m(0)^2 + e_o(0)^2 + |k_p| \Theta_{\max}^2) c_{11}$$

where c_{11} is independent of γ and g_θ , and is only a function of the initial conditions of the system and the fixed design parameters.

Remark 11: Note that

$$\lim_{\ell \rightarrow -\infty} \frac{\ell^2}{g_\theta^2} = 1.$$

Thus for large $|\ell|$ the truncated L-2 norm of $\dot{\theta}$ is simply a function of the initial conditions of the system and the tuning parameter η .

Remark 12: The similarity of the bounds in Theorem 10 to those in Theorem 5 implies that the same bounds on frequencies and corresponding amplitudes of the overall adaptive system hold here in the CMRAC-CO case as well.

C. ROBUSTNESS OF CMRAC-CO TO NOISE

As mentioned earlier, the benefits of the CMRAC-CO is the use of the observer state x_o rather than the actual plant state x . This implies that the effect of any measurement noise on system performance can be reduced. This is explored in this section and Section V-D.

Suppose that the actual plant dynamics is modified from (1) as

$$\dot{x}_a(t) = a_p x_a(t) + k_p u(t), \quad x_p(t) = x_a(t) + n(t) \quad (65)$$

where $n(t)$ represents a time varying disturbance. For ease of exposition, we assume that $n \in \mathcal{C}_1$.

This leads to a set of modified error equations

$$\begin{aligned} \dot{e}_m(t) &= (a_m + \ell) e_m + k_p \bar{\theta}(t) x_o + k_p \theta^* e_o + \xi(t) \\ \dot{e}_o(t) &= (a_m + \ell - k_p \theta^*) e_o - k_p \bar{\theta}(t) x_o - \xi(t) \end{aligned} \quad (66)$$

where

$$\xi(t) \triangleq \dot{\eta}(t) - a_p \eta(t) \quad (67)$$

Theorem 11: The adaptive system with the plant in (65), the controller defined by (52), the update law in (55) with the reference model as in (2), and ℓ chosen such that $a_m + \ell + 2|k_p| |\theta^*| < 0$, all trajectories are bounded and

$$\begin{aligned} V(t) &\leq \frac{1}{2} (e_m(0)^2 + e_o(0)^2) e^{-2|g_n|t} \\ &+ \frac{|k_p|}{\gamma} \Theta_{\max}^2 + \frac{1}{4|g_n|^2} \|\xi(t)\|_{L_\infty}^2. \end{aligned} \quad (68)$$

where

$$g_n \triangleq a_m + \ell + 2|k_p| |\theta^*|. \quad (69)$$

Proof: Taking the time derivative of V in (57) results in

$$\begin{aligned} \dot{V} &\leq g_n (\|e_m\|^2 + \|e_o\|^2) - |k_p| \frac{\eta}{\gamma} \epsilon_\theta^2 \\ &+ \|\xi(t)\| \|e_m(t)\| + \|\xi(t)\| \|e_o(t)\|. \end{aligned} \quad (70)$$

completing the square in $\|e_m\| \|n\|$ and $\|e_o\| \|n\|$

$$\begin{aligned} \dot{V} &\leq -|g_n|/2 (\|e_m\|^2 + \|e_o\|^2) - |k_p| \frac{\eta}{\gamma} \epsilon_\theta^2 \\ &- |g_n|/2 (\|e_m\| - 1/|g_n| \|\xi(t)\|)^2 \\ &- |g_n|/2 (\|e_o\| - 1/|g_n| \|\xi(t)\|)^2 \\ &+ 1/(4|g_n|) \|\xi(t)\|^2. \end{aligned}$$

Neglecting the negative terms in lines 2 and 3 from above and the term involving ϵ_θ we have that

$$\dot{V} \leq -|g_n|/2 (\|e_m\|^2 + \|e_o\|^2) + 1/(4|g_n|) \|\xi(t)\|^2,$$

and in terms of V gives us

$$\dot{V} \leq -|g_n| V + \frac{1}{2} \frac{|k_p| |g_n|}{\gamma} (\tilde{\theta}^2 + \bar{\theta}^2) + \frac{1}{4|g_n|} \|\xi(t)\|^2.$$

Using the Gronwall-Bellman Lemma and substitution of $V(t)$ leads to the bound in (68). ■

D. SIMULATION STUDY

For this study a scalar system in the presence of noise is to be controlled with dynamics as presented in (65), where $n(t)$ is a deterministic signal used to represent sensor noise. $n(t)$ is generated from a Gaussian distribution with variance 1 and covariance 0.01, deterministically sampled using a fixed seed at 100 Hz, and then passed through a saturation function with upper and lower bounds of 0.1 and -0.1 respectively. For the CMRAC-CO systems the reference model is chosen as (2) with the rest of the controller described by (51)-(55).

The CMRAC system used for comparison is identical to that in [15]. For CMRAC the reference dynamics are

now chosen as x_m^o in (3), the observer is the same as CMRAC-CO (51).

Further differences arise with the control law being chosen as

$$u = \theta x_p + k^* r$$

The open-loop error $e^o = x_p - x_m^o$ updates the direct adaptive component, with the regressor becoming x_p instead of x_o for both θ and $\hat{\theta}$ update laws:

$$\begin{aligned} \dot{\theta} &= \text{Proj}_{\Omega}(-\gamma \text{sgn}(k_p) e_m^o x_p, \theta) - \eta \epsilon_{\theta} \\ \dot{\hat{\theta}} &= \text{Proj}_{\Omega}(\gamma \text{sgn}(k_p) e_o x_p, \hat{\theta}) + \eta \epsilon_{\theta}. \end{aligned} \quad (71)$$

The complete CMRAC and CMRAC-CO systems are given in Table 1 with the design parameters given in Table 2.

TABLE 1. Test case equations.

Parameter	CMRAC	CMRAC-CO
Reference Dynamics	Equation (3)	Equation (2)
Observer Dynamics	Equation (51)	Equation (51)
Reference Error	$e_m^o = x_p - x_m^o$	$e_m = x_p - x_m$
Observer Error	$e_o = x_o - x_p$	$e_o = x_o - x_p$
Input	$u = \theta x_p + r$	$u = \theta x_o + r$
Update Laws	Equation (71)	Equation (55)

TABLE 2. Simulation parameters.

Parameter	Value
a_p	1
k_p	1
a_m	-1
k_m	1
ℓ	-10
γ	100
η	1

The simulations have two distinct regions of interest, with Region 1 denoting the first 4 seconds, Region 2 denoting the 4 sec to 15 sec range. In Region 1, the adaptive system is subjected to non-zero initial conditions in the state and the reference input is zero. At $t = 4$ sec, the beginning of Region 2, a filtered step input is introduced. Figures 7 and 8 illustrate the response of the CMRAC-CO adaptive system over 0 to 15 seconds, with x_m , x , and e_m indicated in Figure 7, and u , $\Delta u/\Delta t$, θ and $\hat{\theta}$ indicated in Figure 8. The addition of sensor noise makes the output x_p not differentiable and therefore we use the discrete difference function Δ to obtain the discrete time derivative of the control input, where

$$\frac{\Delta u}{\Delta t} \triangleq \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i}, \quad t_{i+1} - t_i = 0.01.$$

In both cases, the resulting performance is compared with the classical CMRAC system. The first point that should be noted is a satisfactory behavior in the steady-state of the CMRAC-CO adaptive controller. We note a significant difference between the responses of CMRAC-CO and CMRAC systems, which pertains to the use of filtered regressors in CMRAC-CO. An examination of $\Delta u/\Delta t$ in Figure 8 clearly illustrates the advantage of CMRAC-CO.

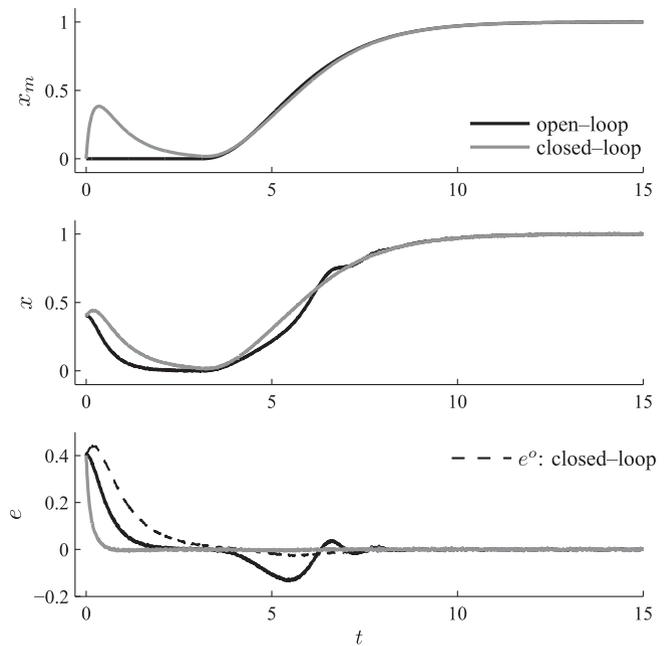


FIGURE 7. (Top) reference model trajectories x_m , (middle) state x , and (bottom) model following e .

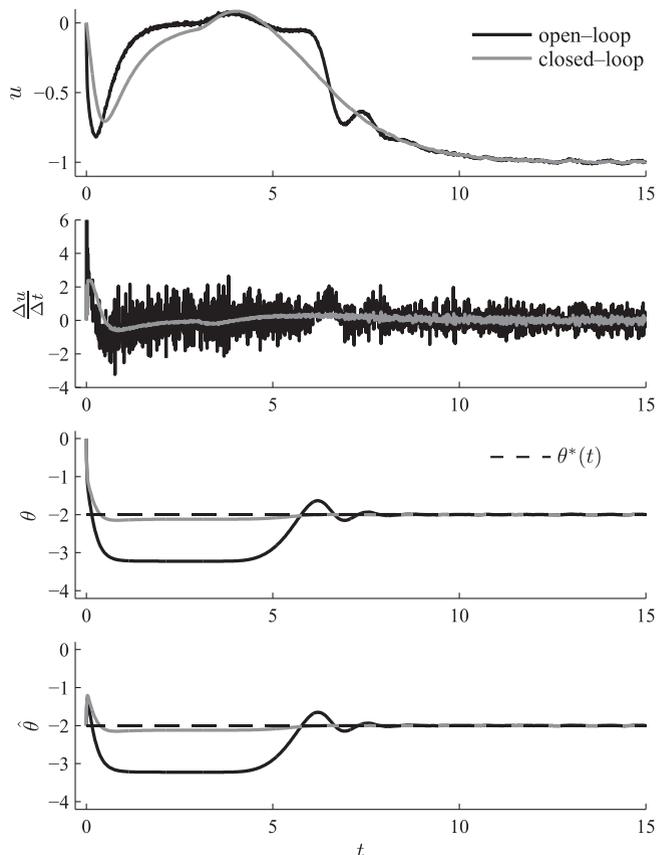


FIGURE 8. (Top) Control input u , (middle-top) discrete rate of change of control input $\Delta u/\Delta t$, (middle-bottom) adaptive parameter $\theta(t)$ and (bottom) adaptive parameter $\hat{\theta}(t)$.

E. COMMENTS ON CMRAC AND CMRAC-CO

As discussed in the Introduction, combining indirect and direct adaptive control has always been observed to produce

desirable transient response in adaptive control. While the above analysis does not directly support the observed transient improvements with CMRAC, we provide a few speculations below: The free design parameter ℓ in the identifier is typically chosen to have eigenvalues faster than the plant that is being controlled. Therefore the identification model following error e_i converges rapidly and $\hat{\theta}(t)$ will have smooth transients. It can be argued that the desirable transient properties of the identifier pass on to the direct component through the tuning law, and in particular $\epsilon\theta$.

VI. CRMS IN OTHER ADAPTIVE SYSTEMS

While CRMs can be traced to [3] in the context of direct model reference adaptive control, such a closed loop structure has always been present in, adaptive observers, tuning function designs, and in a similar fashion in adaptive control of robots. These are briefly described in the following sections.

A. ADAPTIVE BACKSTEPPING WITH TUNING FUNCTIONS

The control structure presented here is identical to that presented in [18, §4.3], but without nonlinear damping terms. Consider the unknown system

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta^* \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta^* \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \dots, x_{n-1})^T \theta^* \\ \dot{x}_n &= \beta(x)u + \varphi_n(x)^T \theta^* \end{aligned} \quad (72)$$

where θ^* is an unknown column vector, $\beta(x)$ is known and invertible, the φ_i are known, x is the state vector of the scalar x_i and the goal is to have $y = x_1$ follow a desired n times differentiable y_r . The control law propped in [18] is of the form

$$u = \frac{1}{\beta(x)} \left(\alpha_n + y_r^{(n)} \right) \quad (73)$$

with an update law

$$\dot{\theta} = \Gamma Wz \quad (74)$$

where $\Gamma = \Gamma^T > 0$ is the adaptive tuning parameter, z is the transformed state error, and $W = \tau_n(z, \theta)$ with τ_i and the α_i , $1 \leq i \leq n$ defined in (84) in the Appendix along with the rest of the control design. The closed loop system reduces to

$$\dot{z} = A_z(z, \theta, t)z + W(z, \theta, t)\tilde{\theta} \quad (75)$$

where $\tilde{\theta} = \theta - \theta^*$ and

$$A_z = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1 - \sigma_{n-1,n} & -c_n \end{bmatrix}$$

where $\sigma_{ik} = -\frac{\partial \alpha_{i-1}}{\partial \theta} \Gamma w_k$. The c_i are free design parameters that arise in the definition of the α_i as defined in (84) in the

Appendix. Notice that the $-c_i$ act in the same way as the ℓ in the simple adaptive system first presented in the reference model in (2). They act to close the reference trajectories with the plant state. The above system also results in similar L-2 norms for the z error state. Consider the Lyapunov candidate

$$V(z(t), \tilde{\theta}(t)) = \frac{1}{2}z^T z + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \quad (76)$$

which results in a negative semidefinite derivative $\dot{V} \leq -c_0 \|z\|^2$ where $c_0 = \min_{1 \leq i \leq n} (c_i)$. Thus we can integrate $-\dot{V}$ to obtain the following bound on the L-2 norm of z

$$\|z(t)\|_{L_2}^2 \leq \frac{V(0)}{c_0}. \quad (77)$$

It is addressed in [18, §4.4.1] that while it may appear that increasing c_0 uniformly decreases the L-2 norm of z , choosing c_i to be large can result in large $z(t)$. The authors then provide a method for initializing the z dynamics so that $z(0) = 0$. We have already discussed why this may not be possible in a real system.

B. ADAPTIVE CONTROL IN ROBOTICS

The control structure presented here is taken directly from [19, §9.2]. Consider the dynamics of a rigid manipulator

$$H(q)\ddot{q} + C(q, \dot{q}) + g(q) = \tau \quad (78)$$

where q is the joint angle, and τ is the torque input. It is assumed that the system can be parameterized as

$$Y(q, \dot{q}, \ddot{q}_r) a = H(q)\ddot{q}_r + C(q, \dot{q})\dot{q}_r + g(q) \quad (79)$$

where q_r is a twice differentiable reference signal, Y is known and a is an unknown vector. The control law is chosen as

$$\tau = Y\hat{a} - k_d s \quad \text{and} \quad \dot{\hat{a}} = -\Gamma Y^T s. \quad (80)$$

Then, defining the desired dynamics trajectory as q_d , the reference dynamics of the system are created by

$$\dot{q}_r = \dot{q}_d - \lambda \tilde{q} \quad (81)$$

where $\tilde{q} = q - q_d$ and

$$s = \dot{q} - \dot{q}_r = \dot{q} + \lambda \tilde{q}. \quad (82)$$

The stability of the above system can be verified with the following Lyapunov candidate,

$$V = \frac{1}{2} \left(s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a} \right).$$

Differentiating and using the property that $\dot{H} = C + C^T$ we have that

$$\dot{V} = -s^T k_d s. \quad (83)$$

We note that λ has a similar role in this control structure as the ℓ in the CRM. The desired trajectory is q_d (like x_m^o in our examples), however the adaptive parameter is updated by the composite variable s instead of directly adjusted by the true reference error.

We now conjecture as to why closed-loop reference models have not been studied in direct adaptive control until recently. In the two cases of nonlinear adaptive control, closed reference trajectory errors are used to update the adaptive controller. This is performed in the tuning function approach through the selection of the c_i and in the adaptive robot control example with λ and the creation of the composite variable s . In both cases the stability of the system necessitates the introduction of these variables. In contrast, in model reference adaptive control, stability is derived from the inherent stability of the reference model and hence any addition of new variables becomes superfluous. When no reference model is present, closing the loop on the reference trajectory becomes necessary. With the recent focus on improving transients in adaptive systems, CRM now has a role in MRAC. And as pointed out in this paper, improved transients can result with CRM without introducing peaking by choosing the ratio $|\ell|/\gamma$ carefully.

VII. CONCLUSION

An increasingly oscillatory response with increasing adaptation gain is a transient characteristic that is ubiquitous in all adaptive systems. Recently, a class of adaptive systems has been investigated with closed-loop reference models where such oscillatory response can be minimal. In this paper, a detailed analysis of such adaptive systems is carried out. It is shown through the derivation of analytical bounds on both states of the adaptive system and on parameter derivatives that a phenomenon of peaking can occur with CRMs and that this phenomenon can be curtailed through a combination of design and analysis, with the peaking exponent reduced from 0.5 to zero. In particular, it is shown that bounds on the parameter derivatives can be related to bounds on frequencies and corresponding amplitudes, thereby providing an analytical basis for the transient performance. This guarantees that the resulting adaptive systems have improved transient characteristics with reduced oscillations even as the adaptation gains are increased. CRMs are shown to be implicitly present in other problems including composite control, adaptive nonlinear control and a class of problems in robotics.

APPENDIX

TUNING FUNCTION PARAMETER DEFINITIONS

$$\begin{aligned}
 z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \\
 \alpha_i &= -z_{i-1} - c_i z_i - w_i^T \theta \\
 &\quad + \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_k} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) \\
 &\quad + \frac{\partial \alpha_{i-1}}{\partial \theta} \Gamma \tau_i + \sum_{k=2}^{i-1} \frac{\partial \alpha_{k-1}}{\partial \theta} \Gamma w_i z_k \\
 \tau_i &= \tau_{i-1} + w_i z_i \\
 w_i &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k
 \end{aligned} \tag{84}$$

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