Proceedings of the 31st Conference
on Decision and Control
Tucson, Arizona • December 1992

A COMPUTATIONAL ALGORITHM FOR SQUARING-UP PART I: ZERO INPUT-OUTPUT MATRIX

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Abstract

In this paper, some preliminary results on the problem of squaring-up a non-square linear multivariable system are presented. It is shown that for the case considered here, the squaring-up problem can be transformed into a state feedback problem. An illustrative numerical example is included to illustrate the results developed.

1. Introduction

In certain applications, e.g., LQG/LTR [1] of non-square systems, one possible approach to design controllers is to make the system square. This is accomplished by finding additional *pseudo-inputs* or *pseudo-outputs* such that the resulting square system is a minimum-phase system (with transmission zeros at desired locations in left half plane). It should be pointed out that the problem of squaring can also be solved by squaring the system down such that the resulting square system has minimum-phase [2]. However, it is well known that this is equivalent to output feedback compensation and may typically require dynamic compensation, thereby increasing the order and complexity of the system.

If one has complete freedom in selecting the locations of actuators and/or sensors, then under certain mild assumptions, it is possible to ensure that the transmission zeros of the system can be assigned at arbitrary locations in complex plane [3], [4]. However, the freedom of selection of actuators or sensors in squaring-up a non-square system is considerably reduced. Formally, the problem addressed here may be stated as follows:

"Given the state matrix ($A \in R^{n\times n}$, system dynamics), the input matrix ($B \in R^{n\times m}$, location of actuators), the output matrix ($C \in R^{p\times n}$, location of the sensors) and the input-output interaction matrix $D \in R^{p\times m} = O$, $p \neq m$. Determine a pseudo-output matrix $\hat{C} \in R^{(m-p)\times n}$ if $p < m$, such that the resulting square system has its zeros in the left half plane."

Of course, the problem of determining a pseudo-input matrix $\hat{B}$, when $m < p$ is the dual of the above problem and can be easily solved. In next two sections, we determine the conditions under which the above problem can be solved and develop a computational scheme for its solution.

2. Main Results

In the sequel, we will discuss only the case when the system is *fat*, i.e., number of inputs is more than number of outputs. The case of *tall* systems is true by duality. Although the results are general, due to lack of space, we will consider only the cases where $D = O$. It will be assumed that only the elements of C need to be augmented, the input-output interaction matrix D is to remain a null matrix. The general case will be presented in a more complete version of this paper. The following assumptions will be made on the system:

1) (A, B) is a controllable pair and B has full column rank ($= m$),
2) rank$(CB) = p$ (same as the number of outputs of the system).

Provided that Assumptions 1 and 2 above are satisfied, theoretically it is always possible to transform the system (A, B, C), by means of orthogonal state coordinate transformations, to the following form:

$$\hat{S}(\lambda) = \left[\begin{array}{cc|c} A_{11} - \lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-m} & O \\ \hline C_{11} & C_{12} & O \end{array}\right]. \tag{2.1}$$

Now, by Assumption 2), the rank of C_{11} is p. Denoting the pseudo-output matrix to be determined by $[C_{21} \quad C_{22}]$ such that rank of $\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$ is m, the system matrix can be written as

$$S(\lambda) \triangleq \left[\begin{array}{cc|c} A_{11} - \lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-m} & O \\ \hline C_{11} & C_{12} & O \\ C_{21} & C_{22} & O \end{array}\right] \tag{2.2}$$

Let, $[C_1 \quad C_2] \triangleq \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, then,

$$\begin{aligned} \text{rank}(S(\lambda)) &= \text{rank}\left[\begin{array}{cc|c} A_{11} - \lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-m} & O \\ \hline C_1 & C_2 & O \end{array}\right] \\ &= \text{rank}\left(S(\lambda) \left[\begin{array}{cc|c} I_m & -C_1^{-1}C_2 & O \\ O & I_{n-m} & O \\ \hline O & O & I_m \end{array}\right]\right) \\ &= \text{rank}\left[\begin{array}{cc|c} A_{11} - \lambda I_m & \times & B_1 \\ A_{21} & A_{22} - A_{21}C_1^{-1}C_2 - \lambda I_{n-m} & O \\ \hline C_1 & O & O \end{array}\right]. \end{aligned} \tag{2.3}$$

Since rank of C_1 is m by construction, clearly,

$$\text{rank}(\mathcal{S}(\lambda)) = 2m + \text{rank}[\lambda I_{n-m} - A_{22} + A_{21}C_1^{-1}C_2]. \quad (2.4)$$

Further, $\text{rank}(\mathcal{S}(\lambda)) < n + m$ at all eigenvalues of the matrix $[A_{22} - A_{21}C_1^{-1}C_2]$. Since, A_{22} and $A_{21}C_1^{-1}$ are known matrices, C_2 can be selected such that the matrix $A_{22} - A_{21}C_1^{-1}C_2$ has all its eigenvalues at desired locations in the left half plane. Equivalently, the problem of finding the *augmented* output matrix $[C_1 \quad C_2]$ such that the system (A, B, C, O) is minimum phase, which in turn solves the problem of squaring up a non-square system, can be reduced to solving a state feedback problem. Notice that several excellent numerical techniques exist for the solution of the state feedback problem [5], [6].

3. Computation of C_1 and C_2

Under the assumption that rank of $C_{11} = p$, it is easy to see that C_{21} can be chosen such that the matrix C_1 has full rank. Any C_{21} lying in the null space of C_{11} will accomplish this goal. Numerical algorithms such as singular value decomposition can be employed to determine C_{21}.

Determination of C_2 is not so straightforward. Note that it is required that the matrix $A_{22} - A_{21}C_1^{-1}C_2$ have all its eigenvalues at desired locations. To see how this may be accomplished, let us write $C_2 := \hat{C}_2 + \check{C}_2$, where

$$\check{C}_2 = \begin{bmatrix} C_{12} \\ O_{(m-p)\times(n-m)} \end{bmatrix} \quad \text{and} \quad \hat{C}_2 = \begin{bmatrix} O_{p\times(n-m)} \\ C_{22} \end{bmatrix}, \quad (3.1)$$

where the subscript of O denotes its dimension and $C_{22} \in R^{(m-p)\times(n-m)}$.

Next, let $\tilde{A}_{22} \triangleq A_{22} - A_{21}C_1^{-1}\check{C}_2$. Then, the problem of determining $\hat{C}_{22}$ reduces to finding a state feedback matrix C_{22} such that the matrix $\tilde{A}_{22} - A_{21}C_1^{-1}\hat{C}_2$ has desired eigenvalues, where $\hat{C}_2 = \begin{bmatrix} O_{p\times(n-m)} \\ C_{22} \end{bmatrix}$.

The above problem can be solved provided the subsystem $(A_{22},\ A_{21})$ is controllable. Note that the original system is assumed to be controllable. It is well known that for a controllable system

$$\text{rank} \left[\begin{array}{cc|c} A_{11} - \lambda I_m & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I_{n-m} & O \end{array} \right] = n. \quad (3.2)$$

Knowing that rank $B_1 = m$ by assumption, the rank of $[A_{21},\ A_{22} - \lambda I_{n-m}]$ must be $n-m$. Equivalently (A_{22}, A_{21}) is a controllable pair. Therefore, it is always possible to find a C_{22} and hence $\hat{C}_2$ such that

$$\mathcal{S}(\lambda) = \left[\begin{array}{cc|c} A_{11} - \lambda I & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I & O \\ \hline C_{11} & C_{12} & O \\ C_{21} & C_{22} & O \end{array} \right] \quad (3.3)$$

has its zeros at the desired location in the left half plane. It should be emphasized that if the given system did possess any transmission zeros, they can be reassigned to the original locations (if required) by state feedback represented by $\tilde{A}_{22} - A_{21}C_1^{-1}\hat{C}_2$.

4. Numerical Example

For the purpose of illustration, consider a 5-th order, 3 inputs, 2 output system (A, B, C) where

$$A = \begin{bmatrix} 1 & 3 & 2 & 3 & 2 \\ 1 & 1 & 4 & 3 & 5 \\ 4 & 3 & 1 & 4 & 5 \\ 1 & 4 & 1 & 3 & 2 \\ 1 & 2 & 3 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 \\ 5 & 4 \\ 0 & 0 \\ 4 & 4 \\ 2 & 5 \end{bmatrix} \quad \text{and}$$

$$C = [3 \quad 3 \quad 4 \quad 0 \quad 3].$$

The given non-square system has no transmission zeros. On applying the technique proposed in this paper, it was found that to assign the transmission zeros of the squared system at $\{-1,\ -2,\ -3\}$, the *pseudo* output $[C_{21}\ C_{22}]$ (in the transformed coordinates) was: $[0.0000e^{+00},\ 1.0000e^{+00},\ -4.5690e^{-02},\ 4.8522e^{-01},\ -1.8061e^{-01}]$. Finally, in the coordinates of the original system, the second row of the output matrix that will assign the transmission zeros at -1, -2 and -3 was found to be $[-2.3327e^{-01},\ -7.2575e^{-01},\ -4.5690e^{-02},\ 2.9193e^{-01},\ 7.7570e^{-01}]$.

5. Concluding Remarks

Even if the given non-square system system has no transmission zeros, when squared, the resulting system will possess some transmission zeros. This paper presented a numerical approach for squaring-up a system. The proposed approach ensures that the resulting squared system can retain the existing zeros where they were located and assign any additional zeros at desired locations.

References

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